Graduate Algebra
Homework 7

Fall 2014

Due 2014-10-29 at the beginning of class

1. Let N be a normal subgroup of G. If N and G/N are solvable, show that G is solvable.

   Proof. The solvability condition implies that $N = N_0 \rhd N_1 \rhd \ldots \rhd N_k = 1$ with N_i/N_{i+1} abelian and $G/N = H_0 \rhd H_1 \rhd \ldots \rhd H_l = 1$ with H_i/H_{i+1} abelian. We proved in class that $H_i' = \{g \in G | g+N \in H_i\}$ is a normal subgroup of G and $N \subset H_i'$.

   Now the third isomorphism theorem for groups shows that $H_i'/H_{i+1}' \cong (H_i'/N)/(H_{i+1}'/N)$ and the first isomorphism theorem shows that $H_i'/N \cong H_i$. Thus H_i'/H_{i+1}' is abelian. Thus we get the composition series
   $$G \rhd H_1' \rhd \ldots H_l' = N \rhd N_1 \rhd \ldots \rhd N_k = 1$$
   and successive quotients are abelian. Thus G is solvable. □

2. Show that D_{2n} is nilpotent iff n is a power of 2.

   Proof. I claim by induction that $D_{2n}^{(k)}$ defined as $D_{2n}^{(0)} = D_{2n}$ and $D_{2n}^{(k+1)} = [D_{2n}^{(k)}, D_{2n}]$ is $D_{2n}^{(k)} = \langle R^{2^k} \rangle$ for $k \geq 1$. Indeed, we saw in class that $[D_{2n}, D_{2n}] = \langle R^2 \rangle$, giving the base case. For the inductive step enough to note that $[R^{2^k i}, R^j] = 1$ and $[R^{2^k i}, FR^j] = R^{2^k i} F R^j R^{-2^k i} R^{-j} F = R^{2^k i + 2^k i} = R^{2^{k+1} i}$.

   Thus D_{2n} is nilpotent iff $\langle R^{2^k} \rangle = 1$ for k large enough. Writing additively, need to find n such that $2^k(\mathbb{Z}/n\mathbb{Z}) \subset \mathbb{Z}/n\mathbb{Z}$ is trivial for k large enough.

   Write $n = 2^a m$ with m odd. Then
   $$2^k(\mathbb{Z}/2^a m\mathbb{Z}) = \begin{cases} \mathbb{Z}/2^{a-k} m\mathbb{Z} & k \leq a \\ 2^{k-a}\mathbb{Z}/m\mathbb{Z} & k > a \end{cases}$$
   Since m is odd then $x \mapsto 2^{k-a}x$ is an automorphism of $\mathbb{Z}/m\mathbb{Z}$ and so $2^{k-a}\mathbb{Z}/m\mathbb{Z} = \mathbb{Z}/m\mathbb{Z}$ which implies that $D_{2n}^{(k)} = \mathbb{Z}/m\mathbb{Z}$ for k large enough. Thus D_{2n} is nilpotent iff $n = 2^a$. □

3. Let $I = \mathbb{Z}_{n \geq 1}$ with partial order $m \leq n$ iff $m \mid n$.

   (a) Show that I is a directed set.

   (b) Let $G_n = \mathbb{Z}$ and for $m \mid n$ let $\iota_{m,n}(x) = xn/m$. Show that (G_n) is a direct system of groups.

   (c) Show that $\varinjlim G_n \cong \mathbb{Q}$.

   Proof. (a): If $m, n \in \mathbb{Z}_{\geq 1}$ then $m, n \leq mn$ so I is a directed set.

   (b): Suppose $m \leq n \leq p$. Then $\iota_{n,p} \circ \iota_{m,n}(x) = \iota_{n,p}(xn/m) = xn/mp/n = xp/m = \iota_{m,p}(x)$. This shows that (G_n) is a direct system of groups.

(c): Consider $f : \mathbb{Q} \to \varinjlim G_n$ sending m/n to $m \in \mathbb{Z} = G_n$. We need to check that this is well-defined. But $m/n = p/q$ iff $m/n = mk/nk = pl/ql = p/q$ with $mk = pl$ and $nk = ql$. But $f(m/n) = m \in G_n$, $f(mk/nk) = mk \in G_{nk}$ and this is simply $\iota_{n,nk}(m \in G_n)$ which is equal to $m \in G_n$ in $\varinjlim G_n$. Thus $f(m/n) = f(mk/nk) = f(pl/ql) = f(p/q)$ and so f is well-defined. Moreover, $f(m/n + p/q) = f((mq + np)/(nq)) = mq + np \in G_{nq}$ while $f(m/n) + f(p/q) = (m \in G_n) + (p \in G_q) = (mq \in G_{nq}) + (np \in G_{nq}) = (mq+np) \in G_{nq}$ and so f is a homomorphism. Finally, f is injective ($m/n = 0$ iff $m = 0$) and surjective as $x \in \varinjlim G_n$ is of the form $m \in G_n$ for some m and n and so $x = f(m/n)$. □

4. Show that $\mathbb{Z}_p$ is torsion-free, i.e., there is no element $x \in \mathbb{Z}_p$ such that $mx = 0$ for some nonzero integer m. [Here $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$.]

   Proof. $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ and let $x = (x_p, x_{p^2}, \ldots) \in \mathbb{Z}_p$ with $x_{p^k} \in \mathbb{Z}/p^k\mathbb{Z}$ such that $x_{p^{k+1}} \equiv x_{p^k} \pmod{p^k}$. Suppose $mx = 0$ for some $m \in \mathbb{Z} - 0$. Then $mx_{p^k} = 0$ for all k and so $mx_{p^k} \equiv 0 \pmod{p^k}$ for all k. Choose representatives $y_{p^k} \in \mathbb{Z}$ for $x_{p^k} \in \mathbb{Z}/p^k\mathbb{Z}$. Then $p^k \mid my_{p^k}$ for all k. Write $m = p^a n$ with $p \nmid n$. Then $p^k \mid p^a n y_{p^k}$ for all k and so $p^{k-a} \mid ny_{p^k}$ for $k > a$. Since $p \nmid n$ this implies that $p^{k-a} \mid y_{p^k}$ and so $y_{p^k} \equiv 0 \pmod{p^{k-a}}$. But $y_{p^k} \equiv y_{p^{k-a}} \pmod{p^{k-a}}$ and so $x_{p^{k-a}} = 0$ for all $k > a$ giving $x = 0$. □

5. Show that every open subgroup of a topological group is closed in the topology. Deduce that every open subgroup of a profinite group is compact.

   Proof. Note that $G = \sqcup_{g \in G/H} gH$ and so $H = G \setminus \sqcup_{g \in G/H - \{1\}} gH$. Since gH is open, so is the disjoint union and so H is closed.

   A profinite group is compact and a closed subset of a compact set is compact. □

6. Let $(G_u)_{u \in I}$ be a direct system of finite groups with homomorphisms $\iota_{u,v} : G_u \to G_v$ for $u \leq v$ and $\iota_u : G_u \to G := \varinjlim G_u$. If H is a subgroup of G show that $(H_u)_{u \in I}$ with $H_u = \iota_u^{-1}(H)$ is a direct system of groups with $H = \varinjlim H_u$.

   Proof. Only need to check that $\iota_{u,v}$ sends H_u to H_v. Suppose $h \in H_u$, i.e., $\iota_u(h) \in H$. Then $\iota_v(\iota_{u,v}(h)) = \iota_u(h) \in H$ and so $\iota_{u,v}(h) \in H_v$ by definition.

   Finally, the map $\varinjlim H_u \to H$ sending $h \in H_u$ to $\iota_u(h)$ is well-defined by the same computation as above. It is also a homomorphism as each ι_u is a homomorphism. If $\iota_u(h) = 1$ then $h \in \ker \iota_u \cap G_u = 1 \in G_u$ so ι_u is injective. Finally, if $h \in H$ then $h \in G_u$ for some u and $\iota_u(h \in G_u) = h \in H$ so the map is also surjective. □

7. Let G be a topological group and $\widehat{G}$ its Pontryagin dual with the dual topology.
   (a) If G is compact show that $\widehat{G}$ has the discrete topology.
   (b) Compute the Pontryagin dual of $\mathbb{R}/\mathbb{Z}$.

Proof. (a): Since translation by a group element is continuous it suffices to show that $f \equiv 1$ sending $g \in G$ to $1 \in S^1$ gives an open set $\{f\}$. Take $K = G$ compact and $U \subset S^1$ the open right semicircle on S^1. What is $W(K,U) = W(G,U)$? It consists of continuous $\phi : G \to S^1$ sending G to U. Thus $\phi(G) \subset U$ is a group. Choose $z = e^{i\theta} \in \phi(G)$ with $\theta \in (-\pi/2, \pi/2)$. If $\theta \neq 0$, then for $n = \lceil \frac{\pi}{2\theta} \rceil$ we have $z^n = e^{in\theta} \notin U$ but $z^n \in \phi(G)$, contradicting the choice of G. Thus $\phi(G) = 1$ and so $\phi = f$ which implies that $W(K,U) = \{f\}$ is open as desired.

(b): The projection map $p : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is continuous (by definition) and for every $f \in \widehat{\mathbb{R}/\mathbb{Z}}$ the composite $f \circ p : \mathbb{R} \to S^1$ will again be continuous. In class we determined that $\widehat{\mathbb{R}} \cong \mathbb{R}$ the functions being of the form $f_r(x) = e^{2\pi i r x}$ for $r \in \mathbb{R}$. Thus $f \circ p = f_r$ for some r and since $p(1) = 0$ it must be that $1 = f(0) = f(p(1)) = f_r(1) = e^{2\pi i r}$ and so $r \in \mathbb{Z}$. Reciprocally, the map f_r is of the form $f \circ p$ iff $\mathbb{Z} \subset \ker f_r$ so if $r \in \mathbb{Z}$ then this is satisfied and $x \mapsto e^{2\pi i r x}$ for $r \in \mathbb{Z}$ is in $\widehat{\mathbb{R}/\mathbb{Z}}$. □