Graduate Algebra
Homework 8

Fall 2014

Due 2014-11-12 at the beginning of class

Throughout this problem set R is a commutative ring. Recall that $\mathfrak{p} \subset R$ is a prime ideal if $\mathfrak{p} \neq R$ and $R/\mathfrak{p}$ is an integral domain, and $\mathfrak{m} \subset R$ is a maximal ideal if $\mathfrak{m} \neq R$ and $R/\mathfrak{m}$ is a field.

1. Let R be a commutative ring.
   (a) If $\mathfrak{p}$ is a prime ideal of R show that $\mathfrak{p}[X] \subset R[X]$ is a prime ideal. Is $\mathfrak{m}[X] \subset R[X]$ a maximal ideal for a prime ideal $\mathfrak{m}$ of R?
   (b) Suppose $\sqrt{-5} \in R$. Show that $(2, 1+\sqrt{-5})(3, 1-\sqrt{-5}) = (1-\sqrt{-5})$ as ideals.
   (c) Let $I \subset R$ be an ideal. Show that there exists a bijection between the set of all/prime/maximal ideals of R containing I and the set of all/prime/maximal ideals of R/I.

2. Let R be a commutative ring. A **minimal prime ideal** in R is a prime ideal $\mathfrak{p}$ such that if $\mathfrak{q} \subset \mathfrak{p}$ is an ideal of R then either $\mathfrak{q} = \mathfrak{p}$ or $\mathfrak{q} = (0)$. Show that every prime ideal $\mathfrak{p}$ contains a nonzero minimal prime ideal. [Hint: Zorn's lemma.]

3. (a) Let $I, J, \mathfrak{a} \subset R$ be ideals such that $\mathfrak{a} \subset I \cup J$ show that $\mathfrak{a} \subset I$ or $\mathfrak{a} \subset J$.
   (b) Suppose $\mathfrak{p}$ is a prime ideal of R and $\mathfrak{a}_1, \ldots, \mathfrak{a}_n \subset R$ are ideals such that $\cap \mathfrak{a}_i \subset \mathfrak{p}$. Show that $\mathfrak{a}_i \subset \mathfrak{p}$ for some i.

4. Let $\mathfrak{a}, \mathfrak{b} \subset R$ be ideals. Define the **ideal quotient** $(\mathfrak{a} : \mathfrak{b}) = \{x \in R | x\mathfrak{b} \subset \mathfrak{a}\}$.
   (a) Show that $(\mathfrak{a} : \mathfrak{b})$ is an ideal of R.
   (b) Show that $(\mathfrak{a} : \mathfrak{b})\mathfrak{b} \subset \mathfrak{a} \subset (\mathfrak{a} : \mathfrak{b})$ and that if $\mathfrak{c}$ is another ideal then $((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}) = (\mathfrak{a} : \mathfrak{b}\mathfrak{c})$.
   (c) If $m, n \in \mathbb{Z} - 0$ compute $((m) : (n))$ as an ideal of $\mathbb{Z}$.
   (d) Compute $((2, X) : (3, X))$, $((6, X) : (2, X))$ and $((6) : (3, X))$ in $\mathbb{Z}[X]$.

5. (a) Show that $P(X) = a_0 + a_1 X + a_2 X^2 + \cdots \in R[\![X]\!]$ is invertible if and only if $a_0 \in R^\times$.
   (b) Show that in any commutative ring the sum of a unit and a nilpotent is a unit.
   (c) Show that $P(X) = a_0 + a_1 X + \cdots + a_n X^n \in R[X]$ is invertible if and only if $a_0 \in R^\times$ and $a_1, \ldots, a_n$ are nilpotent. [Hint: If $g(X) = b_0 + b_1 X + \cdots + b_m X^m$ is its inverse show that $a_n^{r+1} b_{m-r} = 0$ for all $0 \leq r \leq m$ by induction. Then use the previous part.]
   (d) Show that $P(X)$ is nilpotent if and only if $a_0, \ldots, a_n$ are all nilpotent.
   (e) Show that in $R[X]$ the nilradical is the same as the Jacobson radical.
   (f) Compute $\sqrt{(xy, y^3)}$ in $\mathbb{C}[x, y]$ and $\sqrt{(108)}$ in $\mathbb{Z}$.