Throughout this problem set $R$ is a commutative ring. Recall that $p \subset R$ is a prime ideal if $p \neq R$ and $R/p$ is an integral domain, and $m \subset R$ is a maximal ideal if $m \neq R$ and $R/m$ is a field.

1. Let $R$ be a commutative ring.

(a) If $p$ is a prime ideal of $R$ show that $p[X] \subset R[X]$ is a prime ideal. Is $m[X] \subset R[X]$ a maximal ideal for a prime ideal $m$ of $R$?

(b) Suppose $\sqrt{5} \in R$. Show that $(2,1+\sqrt{5})(3,1-\sqrt{5}) = (1-\sqrt{5})$ as ideals.

(c) Let $I \subset R$ be an ideal. Show that there exists a bijection between the set of all/prime/maximal ideals of $R$ containing $I$ and the set of all/prime/maximal ideals of $R/I$.

Proof. (a): $R[X]/p[X] \cong (R/p)[X]$. $p$ is prime so $R/p$ is a domain so $(R/p)[X]$ is a domain so $p[X]$ is a prime. But $(R/p)[X]$ is never a field so $m[X]$ is not maximal.

(b): Let $\alpha = \sqrt{5}$.

\[
(2,1+\alpha)(3,1-\alpha) = (6, 2 - 2\alpha, 3 + 3\alpha, 6) = (6, 2 - 2\alpha, 3 + 3\alpha) = (6, 2 - 2\alpha, 5 + \alpha) = (6, 2 - 2\alpha, 1 - \alpha) = (1 - \alpha)
\]

where in line 3 added the last two gens of line 2, in line 4 subtracted last gen from the first. Finally $2-2\alpha = 2(1-\alpha) and 6 = (1+\alpha)(1-\alpha)$ so we conclude what we wanted.

(c): Take $\pi: R \to R/I$. Know that if $J$ is an ideal of $R/I$ then $\pi^*(J) = \pi^{-1}(J)$ is an ideal of $R$ which necessarily contains $I = \ker \pi$. Moreover, $\pi^*$ takes prime ideals to prime ideals. Also $R/\pi^*(J) \cong \text{Im} \overline{\pi}$ and 

$\overline{\pi}$ is surjective since $\pi$ is. Thus $R/\pi^*(J) \cong (R/I)/J$ so if $J$ is maximal then $\pi^*(J)$ is maximal. If $J$ is an ideal of $R$ containing $I$ then $\pi_*(J) = \pi(J)R/I = (J/I)(R/I) = J/I = \pi(J)$ and $R/J \cong (R/I)/(J/I)$ so again $\pi_*(J)$ is prime/maximal if $J$ is prime/maximal. Finally, $\pi^*\pi_*(J) = \pi^{-1}(J)/I = J + I = J and \pi_*\pi^*(J) = \pi\pi^{-1}(J) = J$ so the maps $\pi^*$ and $\pi_*$ are bijections.

2. Let $R$ be a commutative ring. A **minimal prime ideal** in $R$ is a prime ideal $p$ such that if $q \subset p$ is a prime ideal of $R$ then $q = p$. Show that every prime ideal $p$ contains a nonzero minimal prime ideal. [Hint: Zorn’s lemma.]

Proof. Let $S$ be the set of prime ideals $q \subset p$. Since $p \in S$ the set $S$ is nonempty. Partially order the set $S$ wrt inclusion, i.e., $q < q'$ if $q \supset q'$. Suppose $T$ is an ascending chain in $S$. Let $q_T = \cap_{q \in T} q$. Then $q_T$ is an ideal (any intersection of ideals is an ideal). Suppose $xy \in q_T$. Then $xy \in q$ for any $q \in T$ and
so one of $x$ or $y$ is in $q$. Suppose $y \notin qR$. The $y \notin q$ for some $q \in T$ and so $y \notin q'$ for any $q' \in T$ such that $q \supset q'$. This implies that $x \in q'$ and so $x \in qR$. Thus $qR$ is a prime ideal.

We conclude, using Zorn’s lemma, that $S$ has a maximal element and so $p$ contains a minimal prime ideal. \hfill \Box

3. (a) Let $I, J, a \subset R$ be ideals such that $a \subset I \cup J$ show that $a \subset I$ or $a \subset J$.

(b) Suppose $p$ is a prime ideal of $R$ and $a_1, \ldots, a_n \subset R$ are ideals such that $\cap a_i \subset p$. Show that $a_i \subset p$ for some $i$.

**Proof.** (a): Suppose $a$ is not in $I$ or $J$. Then pick $x \in a - I$ and $y \in a - J$ in which case, since $a \subset I \cup J$ we deduce that $x \in J$ and $y \in I$. Now $x + y \in a \subset I \cup J$. If $x + y \in I$ we deduce that $x \in I$ as well, a contradiction. Similarly for $x + y \in J$.

(b): Suppose we can find $x_i \in a_i - p$. Then $\prod x_i \in \cap a_i \subset p$ and so $\prod x_i \in p$. But this cannot be since $p$ is a prime ideal and we’d get that one of the $x_i$ is in $p$. \hfill \Box

4. Let $a, b \subset R$ be ideals. Define the **ideal quotient** $(a : b) = \{x \in R | xb \subset a\}$.

(a) Show that $(a : b)$ is an ideal of $R$.

(b) Show that $(a : b)b \subset a \subset (a : b)$ and that if $c$ is another ideal then $((a : b) : c) = (a : bc)$.

(c) If $m, n \in \mathbb{Z} - 0$ compute $((m) : (n))$ as an ideal of $\mathbb{Z}$.

(d) Compute $((2, X) : (3, X)), ((6, X) : (2, X))$ and $((6) : (3, X))$ in $\mathbb{Z}[X]$.

**Proof.** (a): Pick $x, y \in (a : b)$ and $r \in R$. Then $xb \subset a$ and $yb \subset a$. But $(x + ry)b = xb + ryb \subset a + ra = a$ and so $x + yr \in (a : b)$ so the quotient is an ideal.

(b): If $x \in a$ then $xb \subset a$ immediately as $a$ is an ideal. That $(a : b)b \subset a$ follows from the definition. Now suppose $x \in R$ such that $x\mathfrak{c} \subset (a : b)$. Thus $xb \subset a$ which implies $x \in (a : bc)$. The converse is identical.

(c): See $a \in \mathbb{Z}$ such that $(a)(n) \subset (m), \text{ i.e., } m | na$. Dividing by $(m, n)$ this is equivalent to $m/(m, n) | n/(m, n)a$ and since $m/(m, n)$ and $n/(m, n)$ are coprime it must be that $m/(m, n) | a$. Thus $((m) : (n)) = (m/(m, n))$.

(d): You can do this explicitly but here is a better way. If $I \subset R$ and $a, b$ contain $I$ then $I$ claim that $(a : b)$ is the preimage of $(a/I : b/I)$ under the projection $R \to R/I$. Indeed, if $xb \subset a$ then immediately $xb/I \subset a/I$. If $(x + I)b/I \subset a/I$ it follows that $xb \subset a + I = a$.

First we apply the observation to $I = (X)$. Thus $((2, X) : (3, X))$ is the preimage in $\mathbb{Z}[X]$ of $((2) : (3))$ in $\mathbb{Z} = \mathbb{Z}[X]/(X)$. But this is (2) from the previous part. Thus $((2, X) : (3, X)) = (2, X)$. Similarly $((6, X) : (2, X)) = \pi^{-1}(((6) : (2))) = \pi^{-1}((3)) = (3, X)$.

\[
((6) : (3, X)) = \{P(X)|P(X)(3, X) \subset (6)\} = \{P(X)|P(3Q + XR) \subset 6\mathbb{Z}[X], \forall Q, R\}
\]

In particular this should be true for $Q = 0, R = 1$ so $6 | PX$ which implies $6 | P$. Thus $((6) : (3, X)) \subset (6) \subset ((6) : (3, X))$ so $((6) : (3, X)) = (6)$. \hfill \Box

5. (a) Show that $P(X) = a_0 + a_1X + a_2X^2 + \cdots \in R[X]$ is invertible if and only if $a_0 \in R^\times$.

(b) Show that in any commutative ring the sum of a unit and a nilpotent is a unit.

(c) Show that $P(X) = a_0 + a_1X + \cdots + a_nX^n \in R[X]$ is invertible if and only if $a_0 \in R^\times$ and $a_1, \ldots, a_n$ are nilpotent. [Hint: if $g(X) = b_0 + b_1X + \cdots + b_mX^m$ is its inverse show that $a_r^{n+1}b_{m-r} = 0$ for all $0 \leq r \leq m$ by induction. Then use the previous part.]
(d) Show that $P(X)$ is nilpotent if and only if $a_0, \ldots, a_n$ are all nilpotent.

(e) Show that in $R[X]$ the nilradical is the same as the Jacobson radical.

(f) Compute $\sqrt{(xy, y^2)}$ in $\mathbb{C}[x, y]$ and $\sqrt{\langle 108 \rangle}$ in $\mathbb{Z}$.

Proof. (a) If $Q(X) = b_0 + b_1 X + \cdots$ then

$$P(X)Q(X) = \sum_{i,j} a_i b_j X^{i+j} = \sum_n X^n \sum_{i+j=n} a_i b_j$$

Thus $PQ = 1$ yields iff

$$a_0 b_0 = 1$$
$$a_1 b_0 + a_0 b_1 = 0$$
$$\vdots$$

If $P$ is invertible the first condition implies $a_0 \in R^\times$. Now suppose $a_0 \in R^\times$. Then we can iteratively compute

$$b_n = -a_0^{-1}(a_n b_0 + \cdots + a_1 b_{n-1})$$

and so we can produce $Q$ as an inverse of $P$.

(b) Suppose $x \in R^\times$ and $u^n = 0$. Then

$$\frac{1}{x + u} = \frac{x^{-1}}{1 + xu^{-1}} = x^{-1} \sum_{k \geq 0} (xu^{-1})^k = x^{-1}(1 + (xu^{-1}) + (xu^{-1})^2 + \cdots + (xu^{-1})^{n-1})$$

since $u^n = 0$ the inverse is truncated.

(c) Have

$$P(X)Q(X) = a_n b_n X^{m+n} + (a_n b_{n-1} + a_{n-1} b_m) X^{m+n-1} + \cdots$$

The base case is $r = 0$ and immediately $a_n b_m = 0$ as $PQ = 1$. Suppose $a_n^r b_{m-r+1} = 0$ for some $0 \leq r \leq m$. We'd like to deduce it for $r$. Look at the coefficient of $X^{m+n-r}$ of degree $\geq n$. The coefficient vanishes and so

$$a_n b_{m-r} + a_{n-1} b_{m-r+1} + \cdots = 0$$

and multiplying with $a_n^r$ we get

$$a_n^{r+1} b_{m-r} + \sum_{i} a_{n-i} a_n^r b_{m-(r-i)} = 0$$

The inductive hypothesis implies that only the first term survives and so $a_n^{r+1} b_{m-r} = 0$.

For $r = m$ we deduce that $a_n^{m+1} = 0$ so $a_n$ is nilpotent.

We now show the statement by induction on $\deg P$. If $P(X)$ is invertible then we know that $a_n$ and so also $a_n X^n$ is nilpotent so by the previous part $P(X) - a_n X^n$ is invertible of degree $n - 1$. By the inductive hypothesis we deduce that $a_0$ is invertible and $a_1, \ldots, a_{n-1}$ are nilpotent.

Now suppose $a_0$ is invertible and $a_i$ is nilpotent for $i \geq 1$. Then $a_i X^i$ are nilpotent so $a_1 X + \cdots + a_n X^n$ is nilpotent. Again the previous part then implies that adding the unit $a_0$ gives $P$ is invertible.

(d) If $a_i$ are nilpotent then $a_i X^i$ are and so their sum $P$ is nilpotent. If $P$ is nilpotent then $XP(X)$ is nilpotent so $1 + XP(X)$ is a unit and we can use the previous part.

(e) We know that $\text{Nil}(R[X]) \subset J(R[X])$ so we need that if $P(X)$ is in the Jacobson radical then it is nilpotent. But then for all polynomials $Q$, $1 - PQ$ is a unit. For $Q = X$ we deduce that $a_i$ are all nilpotent so $P$ is nilpotent as desired.
(d): Seek polynomials $P(x, y)$ such that $P(x, y)^n \in (xy, y^3)$ for some $n$. Then $y \mid P^n$ and so $y \mid P$. If $P = yQ$ then $P^3 = y^3Q^3 \in (xy, y^3)$ so $\sqrt{(xy, y^3)} = (y)$.

Seek $n \in \mathbb{Z}$ such that $n^k$ is divisible by $108 = 2^2 \cdot 3^3$. But then $n$ is divisible by 2 and 3 and so by 6. If $n = 6m$ then $n^3 = 216m^3$ is divisible by 108. So $\sqrt{(108)} = (6)$. 

\[\square\]