1. Let $R$ be a PID. Throughout this exercise, $\pi \in R$ represents a prime element, $P(X) \in R[X]$ is an irreducible polynomial and $Q(X) \in R[X]$ is a polynomial whose image in $R/(\pi)[X]$ is irreducible.

(a) Show that $(\pi)$, $(P(X))$ and $(\pi, Q(X))$ are prime ideals of $R[X]$.

(b) Let $p$ be a prime ideal of $R[X]$. Show that $p \cap R$ is either 0 or $(\pi)$.

(c) If $p \cap R = (0)$ show that $p$ is either 0 or some $(P(X))$. [Hint: Show that $p$ gives a prime ideal of $R[X]$ localized at the multiplicative set $R - 0$. What is this localization?]

(d) If $p \cap R = (\pi)$ show that either $p = (\pi)$ or $p = (\pi, Q(X))$ for some $\pi$ and $Q(X)$. [Hint: Look at $R[X]/(\pi)R[X]$.

(e) What are the prime and maximal ideals of $\mathbb{Z}[X]$?

2. Let $R = \mathbb{C}[X,Y]$. [Hint: This is an application of the previous problem.]

(a) Show that the prime ideals of $\mathbb{C}[X,Y]$ are $(0)$, $(P(X,Y))$ for an irreducible $P(X,Y) \in \mathbb{C}[X,Y]$ and $(X - a, Y - b)$ for some $a, b \in \mathbb{C}$. Show that the maximal ideals are $(X - a, Y - b)$.

(b) Show that $p = (Y^2 - X^3 - X^2)$ is a prime ideal and that if $a, b \in \mathbb{C}$ such that $b^2 = a^3 + a^2$ then $p \subset (X - a, Y - b)$.

(c) Let $q = (X - a, Y - b)$. Show that the prime ideals of the localization $R_q$ are: $(0)$, $qR_q$ and $(P(X,Y))R_q$ for any irreducible polynomial $P(X,Y) \in \mathbb{C}[X,Y]$ such that $P(a,b) = 0$.

3. Consider the ring $\mathbb{Z}[\zeta_3]$.

(a) Show that $\mathbb{Z}[\zeta_3]$ is a Euclidean domain. [Hint: Mimick the proof from the $\mathbb{Z}[i]$ case.]

(b) Show that the units are $\mathbb{Z}[\zeta_3]^{\times} = \{\pm 1, \pm \zeta_3, \pm \zeta_3^2\}$. [Hint: Show that $z \in \mathbb{Z}[\zeta_3]$ is a unit iff $|z| = 1$.]

4. Let $R$ be a commutative ring. A commutative ring is said to be reduced if it has no nonzero nilpotent elements.

(a) Suppose that for every prime ideal $p$ the localization $R_p$ is reduced. Show that $R$ is reduced. [Hint: For a given $x$ look at $\{y | xy = 0\}$.]

(b) Show that $R = \mathbb{Z}/6\mathbb{Z}$ is not an integral domain but each localization $R_p$ is an integral domain.

5. Let $R$ be a commutative ring. A proper (i.e., not 0 or $R$) ideal $I$ of $R$ is said to be good if the image of $R^\times \cup 0$ in $R/I$ is all of $R/I$.

(a) Suppose $R$ is a PID with no proper good ideals. Show that $R$ cannot be a Euclidean domain. [Hint: Otherwise, among the proper ideals $I = (a)$ of $R$ choose one with $d(a)$ minimal. Show that $I$ is good.]

(b) You may assume that the ring $R = \mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$ is a PID, that $R^\times = \{\pm 1\}$, and that 2 and 3 are prime in $R$. Show that $R$ is not a Euclidean domain. [Hint: Are there good ideals?]