Graduate Algebra
Homework 9

Fall 2014

Due 2014-11-19 at the beginning of class

1. Let R be a PID. Throughout this exercise, $\pi \in R$ represents a prime element, $P(X) \in R[X]$ is an irreducible polynomial and $Q(X) \in R[X]$ is a polynomial whose image in $R/(\pi)[X]$ is irreducible.
   (a) Show that (π), $(P(X))$ and $(\pi, Q(X))$ are prime ideals of $R[X]$.
   (b) Let $\mathfrak{p}$ be a prime ideal of $R[X]$. Show that $\mathfrak{p} \cap R$ is either 0 or (π).
   (c) If $\mathfrak{p} \cap R = (0)$ show that $\mathfrak{p}$ is either 0 or some $(P(X))$. [Hint: Show that $\mathfrak{p}$ gives a prime ideal of $R[X]$ localized at the multiplicative set $R - 0$. What is this localization?]
   (d) If $\mathfrak{p} \cap R = (\pi)$ show that either $\mathfrak{p} = (\pi)$ or $\mathfrak{p} = (\pi, Q(X))$ for some π and $Q(X)$. [Hint: Look at $R[X]/(\pi)R[X]$.]
   (e) What are the prime and maximal ideals of $\mathbb{Z}[X]$?

2. Let $R = \mathbb{C}[X, Y]$. [Hint: This is an application of the previous problem.]
   (a) Show that the prime ideals of $\mathbb{C}[X, Y]$ are (0), $(P(X, Y))$ for an irreducible $P(X, Y) \in \mathbb{C}[X, Y]$ and $(X - a, Y - b)$ for some $a, b \in \mathbb{C}$. Show that the maximal ideals are $(X - a, Y - b)$.
   (b) Show that $\mathfrak{p} = (Y^2 - X^3 - X^2)$ is a prime ideal and that if $a, b \in \mathbb{C}$ such that $b^2 = a^3 + a^2$ then $\mathfrak{p} \subset (X - a, Y - b)$.
   (c) Let $\mathfrak{q} = (X - a, Y - b)$. Show that the prime ideals of the localization $R_{\mathfrak{q}}$ are: (0), $\mathfrak{q}R_{\mathfrak{q}}$ and $(P(X, Y))R_{\mathfrak{q}}$ for any irreducible polynomial $P(X, Y) \in \mathbb{C}[X, Y]$ such that $P(a, b) = 0$.

3. Consider the ring $\mathbb{Z}[\zeta_3]$.
   (a) Show that $\mathbb{Z}[\zeta_3]$ is a Euclidean domain. [Hint: Mimick the proof from the $\mathbb{Z}[i]$ case.]
   (b) Show that the units are $\mathbb{Z}[\zeta_3]^\times = \{\pm 1, \pm\zeta_3, \pm\zeta_3^2\}$. [Hint: Show that $z \in \mathbb{Z}[\zeta_3]$ is a unit iff $|z| = 1$.]

4. Let R be a commutative ring. A commutative ring is said to be **reduced** if it has no nonzero nilpotent elements.
   (a) Suppose that for every prime ideal $\mathfrak{p}$ the localization $R_{\mathfrak{p}}$ is reduced. Show that R is reduced. [Hint: For a given x look at $\{y | xy = 0\}$.]
   (b) Show that $R = \mathbb{Z}/6\mathbb{Z}$ is not an integral domain but each localization $R_{\mathfrak{p}}$ is an integral domain.

5. Let R be a commutative ring. A proper (i.e., not 0 or R) ideal I of R is said to be **good** if the image of $R^\times \cup 0$ in R/I is all of R/I.
   (a) Suppose R is a PID with no proper good ideals. Show that R cannot be a Euclidean domain. [Hint: Otherwise, among the proper ideals $I = (a)$ of R choose one with $d(a)$ minimal. Show that I is good.]
   (b) You may assume that the ring $R = \mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$ is a PID, that $R^\times = \{\pm 1\}$, and that 2 and 3 are prime in R. Show that R is not a Euclidean domain. [Hint: Are there good ideals?]