Graduate Algebra
Homework 10

Fall 2014

Due 2014-12-03 at the beginning of class

Throughout this problem set R is a commutative ring. **This homework looks longer than usual, but it's mostly extra prose; my solutions are as long as for other homeworks.**

1. Let R be an integral domain and M an R-module.
   (a) If R is a PID is and M is finitely generated and projective show that it is free.
   (b) If M is injective show that M is divisible, i.e., if $r \neq 0$ and $m \in M$ then there exists $n \in M$ such that $m = rn$.

2. A **local** ring is a ring with exactly one maximal ideal. Let R be a commutative ring and M an R-module.
   (a) Let $\mathfrak{m}$ be a maximal ideal of R. Show that R is local with maximal ideal $\mathfrak{m}$ if and only if $R - \mathfrak{m} = R^\times$.
   (b) If $\mathfrak{p}$ is a prime ideal of R show that the localization $R_\mathfrak{p}$ is a local ring. [Hint: Use the definition of local.]
   (c) Let X be a topological space and $x \in X$. Let $R = \{f : U \to \mathbb{R} | f \text{ continuous}, x \in U \text{ open}\}/\sim$ where $\sim$ is the equivalence relation $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$ are equivalent $f \sim g$ is $f|_W = g|_W$ for some open neighborhood W of x. Show that R is a local ring with maximal ideal $\mathfrak{m} = \{f \in R | f(x) = 0\}$. (This shows that X is a so-called locally ringed space which is the fundamental object of most geometric theories.) [Hint: Use part (a).]

3. Let R be a commutative ring and M a module over R.
   (a) If R is an integral domain and $\mathfrak{p} \neq (0)$ is a finitely generated prime ideal of R show that

   $$\mathfrak{p} \supsetneq \dots \supsetneq \mathfrak{p}^n \supsetneq \mathfrak{p}^{n+1} \supsetneq \dots$$

   [Hint: Localize at $\mathfrak{p}$; what is the Jacobson radical of the localization?]
   (b) Let $\mathfrak{p} \subset R$ be as above. Show that $\cap \mathfrak{p}^n = 0$. [Hint: Nakayama for the intersection.]
   (c) (Optional, but immediate from (c)) Deduce that if R is a Noetherian ring and $I \subsetneq R$ is an ideal then $\cap I^n = 0$.

4. Let R be a commutative ring and I a directed set. By a direct system of R-modules we mean a direct system $(M_u)_{u \in I}$ of abelian groups M_u which are also R-modules and such that the transition maps $\iota_{u,v} : M_u \to M_v$ are R-module homomorphisms. By an inverse system of R-modules we mean an inverse system $(N_u)_{u \in I}$ of abelian groups N_u which are also R-modules and such that the projection maps $\pi_{v,u} : N_v \to N_u$ are R-module homomorphisms.
   (a) Show that $\varinjlim M_u$ and $\varprojlim N_u$ (limits as abelian groups) are naturally R-modules.

(b) Let R be a ring and $I \subset R$ is an ideal such that $\cap I^n = 0$.

i. Show that $M_n = R/I^n$ with natural projection maps $\pi_{m,n} : R/I^m \to R/I^n$ for $m \geq n$ is an inverse system.

ii. Let $\widehat{R} = \varprojlim R/I^n$, called the I-adic completion of R with respect to I. Endow each R/I^n with the discrete topology and $\widehat{R}$ with the inverse limit topology inherited from the product topology. Show that ring multiplication in $\widehat{R}$ is continuous, i.e., that $\widehat{R}$ is a topological ring. [Already know that $(R, +)$ is a topological group.]

iii. Show that the map $\iota : R \to \widehat{R}$ sending r to $(r + I, r + I^2, \ldots)$ is injective.

iv. Show that $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ is a topological ring.

(c) Let $R = \mathbb{Z}[X]$, $I = (X)$ and $S = \mathbb{Z} - 0$. Show that $\iota(S)^{-1}\widehat{R} \subsetneq \widehat{S^{-1}R}$ where the RHS completion is with respect to the ideal $S^{-1}I$. [Hint: Recall from class that we computed these completions as an abelian groups.]

(d) Show that $\mathbb{Z}_p$ is a local ring with maximal ideal $p\mathbb{Z}_p$. (This is very general phenomenon, completions at maximal ideals yield local rings.) [Hint: Use 2 (a).]

5. In this exercise all capital letters are modules over a ring and all lower case letters are homomorphisms of modules. By "commutative diagram" I mean that the composition of the maps from one node to another depends only on the nodes and not on the path chosen between the nodes. In both cases the rows are exact complexes of modules. [Hint: This is straightforward "diagram chasing"; use exactness and commutativity to show injective/surjective directly from definitions.]

(a) Suppose

$$\begin{CD} M @>>> N @>>> P @>>> 0 \\ @VfVV @VgVV @VhVV \\ M' @>>> N' @>>> P' @>>> 0 \end{CD}$$

is a commutative diagram with exact rows. If f is surjective and g is injective show that h is injective.

(b) (Optional, but straightforward) Suppose

$$\begin{CD} L @>>> M @>>> N @>>> P @>>> Q \\ @VlVV @V{\cong}VmV @VnVV @V{\cong}VpV @VqVV \\ L' @>>> M' @>>> N' @>>> P' @>>> Q' \end{CD}$$

is a commutative diagram with exact rows. Suppose m and p are isomorphisms, l is surjective and q is injective. Show that n is an isomorphism.