Graduate Algebra
Homework 11

Fall 2014

Due 2014-12-10 at the beginning of class

1. Let R be a commutative ring. In class (Lecture 36, Proposition 8) I showed that if $M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is an exact sequence of R-modules then for any R-module N the sequence $0 \to \mathrm{Hom}_R(M'', N) \xrightarrow{g^*} \mathrm{Hom}_R(M, N) \xrightarrow{f^*} \mathrm{Hom}_R(M', N)$ is also exact. Show the converse: If for any R-module N the Hom sequence is exact then the initial sequence $M' \to M \to M'' \to 0$ must also be exact.

2. Let R be a commutative ring and $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ an exact sequence of R-modules. The exact sequence is said to **split** if $M \cong M' \oplus M''$ such that under this isomorphism f becomes the inclusion $M' \hookrightarrow M' \oplus M''$ and g becomes the projection $M' \oplus M'' \to M''$.
   (a) Show that the exact sequence splits if and only if the map $M \to M''$ admits a section, i.e., there exists an R-module homomorphism $s : M'' \to M$ such that $g \circ s = \mathrm{id}_{M''}$.
   (b) Show that if M'' is a projective R-module then $M \cong M' \oplus M''$.

3. Let K be a field.
   (a) Show that $K[x, y, z]/(xy - z)$ is flat as a module over $K[z]$. [Hint: $K[z]$ is a PID.]
   (b) Let $\pi : K[x]/(x^2) \to K$ be the evaluation at $x = 0$ ring homomorphism. Show that $\pi^*(K)$ is not flat as a module over $K[x]/(x^2)$. [Hint: Look at the injection $(x) \hookrightarrow K[x]/(x^2)$.]

4. Consider $R = K[x^2, x^3]$ as a subring of $S = K[x]$ and consider S as an R-module. Show that the map $f : S \to R \oplus R$ sending $P \mapsto (x^3P, -x^2P)$ is an injective homomorphism but $f \otimes 1 : S \otimes_R S \to (R \oplus R) \otimes_R S$ is not injective. Conclude that S is not flat over R.

5. Let R be a commutative ring and $1 \in S \subset T$ be two multiplicatively closed subsets. Let M be an R-module.
   (a) Show that one gets an R-module homomorphism $S^{-1}M \to T^{-1}M$.
   (b) Show that $T^{-1}R \cong T^{-1}(S^{-1}R)$ and $T^{-1}M \cong T^{-1}(S^{-1}M)$ and deduce that $T^{-1}M \cong S^{-1}M \otimes_{S^{-1}R} T^{-1}R$. [Hint: Apply a suitable theorem from class.]
   (c) Deduce that if $S^{-1}M$ is flat over $S^{-1}R$ then $T^{-1}M$ is flat over $T^{-1}R$. [Hint: Apply a suitable theorem from class.]