Math 80220 Algebrai Number Theory
Problem Set 1

Andrei Jorza

Due Wednesday, Jan 29

Throughout this problem set you will use traces and norms. Recall from that that $\mathrm{Tr}_{L/K}(x) = \sum \sigma(x)$ and $N_{L/K}(x) = \prod \sigma(x)$ where $\sigma : L \to \overline{K}$ are the embeddings of L fixing K. When $L = K(\alpha)$ one may enumerate these embeddings by the roots β of the minimal polynomial of α over K, sending α to β and fixing K.

1. Suppose that m and n are coprime and square-free. Show that $[\mathbb{Q}(\sqrt{m}, \sqrt{n}) : \mathbb{Q}(\sqrt{n})] = 2$ and deduce that

$$\mathrm{Tr}_{\mathbb{Q}(\sqrt{m},\sqrt{n})/\mathbb{Q}(\sqrt{n})}(a + b\sqrt{m} + c\sqrt{n} + d\sqrt{mn}) = 2(a + c\sqrt{n})$$
$$N_{\mathbb{Q}(\sqrt{m},\sqrt{n})/\mathbb{Q}(\sqrt{n})}(a + b\sqrt{m} + c\sqrt{n} + d\sqrt{mn}) = a^2 - b^2m + c^2n - d^2mn + 2(ac - bdm)\sqrt{n}$$

2. Let m, n be coprime square-free integers which are not 1. Let $K = \mathbb{Q}(\sqrt{m}, \sqrt{n})$.

   (a) Suppose $m \equiv 2, n \equiv 3 \pmod 4$. Use traces to show that elements of $\mathcal{O}_K$ are of the form

   $$\frac{a + b\sqrt{m} + c\sqrt{n} + d\sqrt{mn}}{2}$$

   where $a, b, c, d \in \mathbb{Z}$. Use norms to $\mathbb{Q}(\sqrt{n})$ to show that a, c are even while b and d have the same parity and conclude that

   $$\mathcal{O}_K = \mathbb{Z}[\sqrt{m}, \sqrt{n}, \frac{\sqrt{m} + \sqrt{mn}}{2}]$$

   (b) Suppose $m, n \equiv 1 \pmod 4$. Again use traces to show that elements of $\mathcal{O}_K$ are of the form

   $$\alpha = \frac{a + b\sqrt{m} + c\sqrt{n} + d\sqrt{mn}}{4}$$

   where $a, b, c, d \in \mathbb{Z}$ are all of the same parity. Show that

   $$\beta = \frac{1 + \sqrt{m} + \sqrt{n} + \sqrt{mn}}{4} \in \mathcal{O}_K$$

   and that one may write

   $$\frac{r + s\sqrt{m} + t\sqrt{n}}{2} = \alpha - d\beta$$

   with $r, s, t \in \mathbb{Z}$ such that $r + s + t$ even (recall that $(1 + \sqrt{m})/2$ and $(1 + \sqrt{n})/2$ are algebraic integers). Conclude that

   $$\mathcal{O}_K = \mathbb{Z}[\frac{1 + \sqrt{m}}{2}, \frac{1 + \sqrt{n}}{2}, \frac{1 + \sqrt{m} + \sqrt{n} + \sqrt{mn}}{4}]$$

3. Let $m \equiv 1 \pmod 9$ be a square-free integer $\neq 1$ and let $K = \mathbb{Q}(\sqrt[3]{m})$.

(a) Show that $\mathrm{Tr}_{K/\mathbb{Q}}(a+b\sqrt[3]{m}+c\sqrt[3]{m^2}) = 3a$ and $N_{K/\mathbb{Q}}(a+b\sqrt[3]{m}+c\sqrt[3]{m^2}) = a^3+b^3m+c^3m^2-3abcm$.

(b) Show that

$$\delta = \frac{1+\sqrt[3]{m}+\sqrt[3]{m^2}}{3} \in \mathcal{O}_K$$

(c) Use traces to show that elements of $\mathcal{O}_K$ are of the form

$$\alpha = \frac{am + b\sqrt[3]{m} + c\sqrt[3]{m}}{3m}$$

with $a, b, c \in \mathbb{Z}$. Use norms to show that $m \mid b, c$ and so we may in fact write

$$\alpha = \frac{r + s\sqrt[3]{m} + t\sqrt[3]{m^2}}{3}$$

for integers r, s, t.

(d) Use the fact that $\alpha - t\delta \in \mathcal{O}_K$ to conclude that

$$\mathcal{O}_K = \mathbb{Z}[\sqrt[3]{m}, \frac{1+\sqrt[3]{m}+\sqrt[3]{m^2}}{3}]$$

4. The discriminant of a polynomial $f \in K[X]$ with roots $\alpha_1, \ldots, \alpha_n$ is

$$\prod_{i<j}(\alpha_i - \alpha_j)^2 = (-1)^{\binom{n}{2}} \prod_{i \neq j}(\alpha_i - \alpha_j)$$

(a) Show that the discriminant of $f(X) = X^n + pX + q$ is

$$(-1)^{\binom{n}{2}} n^n q^{n-1} + (-1)^{\binom{n-1}{2}} (n-1)^{n-1} p^n$$

(b) For p an odd prime show that the discriminant of $\mathbb{Q}(\zeta_p)$ is $(-1)^{(p-1)/2}p^{p-2}$ and deduce that $\mathbb{Q}(\zeta_p)$ contains $\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})$. [Hint: square root of the discriminant is an element of the field!]