Math 80220 Algebraic Number Theory
Problem Set 7

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due Wednesday, April 30

1. Let $p > 2$ be a prime number and $K = \mathbb{Q}(\zeta_p)$. Recall from the first homework that if $p^* = (-1)^{(p-1)/2}p$ then $\mathbb{Q}(\sqrt{p^*}) \subset K$.

   (a) The group $G = \mathrm{Gal}(K/\mathbb{Q}) \cong (\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic and therefore so is its character group $\widehat{G}$. Denote χ a generator, taking a generator of G to ζ_{p-1}. Show that

   $$\chi^{(p-1)/2}(x) = \left(\frac{x}{p}\right)$$

   (b) Show that if H is the subgroup of $G \cong \mathbb{Z}/(p-1)\mathbb{Z}$ corresponding to $\{0, 2, 4, \ldots, p-3\} \subset \{0, 1, 2, \ldots, p-2\}$ then the fixed subfield is $K^H = \mathbb{Q}(\sqrt{p^*})$. [Hint: Show that there is only one quadratic subfield of K.]

   (c) Show that the characters χ^k and $\chi^{k+(p-1)/2}$ are equal on H and conclude that the characters of $\mathrm{Gal}(\mathbb{Q}(\sqrt{p^*})/\mathbb{Q})$ are 1 and $\left(\frac{\cdot}{p}\right)$. Deduce that

   $$\tau\left(\left(\frac{\cdot}{p}\right)\right) = \sqrt{p^*}$$

   [Hint: For the Gauss sum, use the result from class.]

   (d) If $p \equiv 3 \pmod 4$ show that

   $$L\left(\left(\frac{\cdot}{p}\right), 1\right) = \begin{cases} \dfrac{\pi}{3\sqrt{3}} & p = 3 \\ \dfrac{\pi h_{\mathbb{Q}(\sqrt{-p})}}{\sqrt{p}} & p > 3 \end{cases}$$

   and conclude that

   $$B_{1,\left(\frac{\cdot}{p}\right)} = \begin{cases} -\dfrac{1}{3} & p = 3 \\ -h_{\mathbb{Q}(\sqrt{-p})} & p > 3 \end{cases}$$

   and thus that if $p > 3$ we have

   $$h_{\mathbb{Q}(\sqrt{p^*})} = -\frac{1}{p}\sum_{k=1}^{p}\left(\frac{k}{p}\right)k$$

   (e) If $p \equiv 1 \pmod 4$ and $a + b\sqrt{p}$ is a generator for the unit group $\mathcal{O}^\times_{\mathbb{Q}(\sqrt{p})} = \mathbb{Z}[\frac{1+\sqrt{p}}{2}]^\times$ show that

   $$L\left(\left(\frac{\cdot}{p}\right), 1\right) = \frac{2h_{\mathbb{Q}(\sqrt{p})}|\log|a + b\sqrt{p}||}{\sqrt{p}}$$

   and conclude that

   $$h_{\mathbb{Q}(\sqrt{p})} = -\frac{1}{2|\log|a + b\sqrt{p}||}\sum_{k=1}^{p}\left(\frac{k}{p}\right)\log|1 - \zeta_p^k|$$

2. Let $K = \mathbb{Q}(\sqrt{3})$ and $\chi : K^\times \to \mathbb{C}^\times$ be $\chi(x) = \operatorname{sign} N_{K/\mathbb{Q}}(x)$.

   (a) Show that $u \in \mathcal{O}_K^\times$ if and only if $N_{K/\mathbb{Q}}(u) = 1$ (i.e., no -1 can occur). Show that one may choose a generator $u = a + b\sqrt{3}$ of $\mathcal{O}_K^\times$ such that $a, b > 0$. Deduce that $\mathcal{O}_K^\times = \pm(2+\sqrt{3})^{\mathbb{Z}}$. [Hint: Show that $(a+b\sqrt{3})^k = 2+\sqrt{3}$ would imply that $a \leq 2$ and $b \leq 1$.

   (b) Show that if u is a unit in $\mathcal{O}_K^\times$ then $\chi(xu) = \chi(x)$ and deduce that χ defined a character on the group of ideals of K.

   (c) Show that $2\mathcal{O}_K = (1+\sqrt{3})^2$, $3\mathcal{O}_K = (\sqrt{3})^2$ and if $p > 3$ then p splits in $\mathcal{O}_K$ if and only if $\left(\frac{3}{p}\right) = 1$. Show that if p splits with $p = u\overline{u}$ then $\chi(u) = \left(\frac{p}{3}\right)$.

   (d) Deduce that
$$L(\chi, s) := \prod_{\mathfrak{p}} \left(1 - \frac{\chi(\mathfrak{p})}{||\mathfrak{p}||^s}\right)^{-1} = L\left(\left(\frac{\cdot}{4}\right), s\right) L\left(\left(\frac{\cdot}{3}\right), s\right)$$
   [Hint: Use Euler products and decide how a prime of $\mathbb{Q}$ splits in K.]

3. (Optional) Let G be a finite abelian group and $f : G \to \mathbb{C}$ be any function.

   (a) Show that the vector space V of functions from G to $\mathbb{C}$ has the following sets as bases:

      i. The set $\mathcal{B}_1$ of functions $\phi_g : G \to \mathbb{C}$ as $g \in G$ defined as $\phi_g(h) = \begin{cases} 1 & g = h \\ 0 & g \neq h \end{cases}$.

      ii. The set $\mathcal{B}_2$ of characters $\chi \in \widehat{G}$.

   (b) Show that the linear transformation $T : V \to V$ defined by $(T\phi)(g) = \sum_{h \in G} f(g)\phi(gh)$ has matrix $(f(gh^{-1})_{g,h\in G}$ with respect to the basis $\mathcal{B}_1$ and is diagonal with respect to the basis $\mathcal{B}_2$ and conclude that
$$\det(f(gh^{-1})_{g,h\in G} = \prod_{\chi \in \widehat{G}} \sum_{g \in G} \chi(g) f(g)$$

   (c) Show that the vector subspace $W \subset V$ of functions ϕ such that $\sum_{g \in G} \phi(g) = 0$ has the following sets as bases:

      i. The set $\mathcal{B}_1'$ of functions $\psi_g(h) = \phi_g(h) - 1/|G|$.

      ii. The set $\mathcal{B}_2'$ of characters $\chi \neq 1$.

   (d) Show that the linear transformation T stabilizes W (i.e., $T(W) = W$) and with respect to the basis $\mathcal{B}_1'$ has matrix $(f(gh^{-1}) - f(g))_{g,h\neq 1}$ and is diagonal with respect to $\mathcal{B}_2'$ and deduce that
$$\det(f(gh^{-1}) - f(g))_{g,h\neq 1} = \prod_{\chi \neq 1} \sum_{g \in G} \chi(g) f(g)$$