HW5

1. (a) $x, e^x - 1 \in \mathbb{Q}[[x]]$

so $\mathcal{B}(x) = \dfrac{x}{e^x-1} \in \mathbb{Q}((x))$

but $e^x - 1 = x + \frac{x^2}{2!} + \cdots \in x\,\mathbb{Q}[[x]]^\times$

so $\mathcal{B}(x) \in \mathbb{Q}[[x]]$

also $\mathcal{B}(x) + \frac{x}{2} = \dfrac{x}{e^x-1} + \dfrac{x}{2}$.

$$\mathcal{B}(-x) - \frac{x}{2} = \frac{-x}{e^{-x}-1} - \frac{x}{2} = \frac{x}{1-e^{-x}} - \frac{x}{2}$$

$$= \frac{e^x x}{e^x - 1} - \frac{x}{2} = \frac{x}{e^x-1} + \frac{x}{2}$$

so $\mathcal{B}(x) + \frac{x}{2}$ is even so all coeffs of $x^{2k+1}, k \geq 1$ vanish.

(b)
$$\mathcal{B}(x)e^x = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \sum_{m=0}^{\infty} \frac{x^m}{m!}$$

$$= \sum_{m,n \geq 0} B_n \frac{x^{m+n}}{m!\,n!} = \sum_{n \geq 0} \sum_{k=0}^{n} B_k \frac{x^n}{k!(n-k)!}$$

$$= \sum_{n \geq 0} \frac{x^n}{n!} \sum_{k=0}^{n} B_k \binom{n}{k}$$

$$\mathcal{B}(x)e^x = \frac{x(e^x - 1 + 1)}{e^x - 1} = x + \frac{x}{e^x-1} = x + \mathcal{B}(x)$$

so if $n \geq 2$ then $B_n = \sum_{k=0}^{n} B_k \binom{n}{k}$

(c) from (b) $B_n = -\dfrac{\sum_{k=0}^{n-1} B_k \binom{n+1}{k}}{n+1}$

$\mathcal{B}(x) \bmod x^2$ is $\dfrac{x}{e^x-1} = \dfrac{x}{x+\frac{x^2}{2}+\frac{x^3}{6}} =$

$$\equiv \frac{1}{1+\frac{x}{2}+\frac{x^2}{6}} \equiv 1 - \left(\frac{x}{2}+\frac{x^2}{6}\right) + \left(\frac{x}{2}+\frac{x^2}{6}\right)^2$$

$$= 1 - \frac{x}{2} \quad \text{so} \quad B_0 = 1, \quad B_1 = -\frac{1}{2}.$$

$$B_2 = -\frac{B_0 + 3B_1}{3} = -\frac{1-\frac{3}{2}}{3} = \frac{1}{6}$$

$$B_3 = 0 \qquad (\text{odd})$$

$$B_4 = -\frac{B_0 + 5B_1 + 10B_2 + 10B_3}{5} = -\frac{1-\frac{5}{2}+\frac{10}{6}}{5}$$

$$= -\frac{1}{30}$$

(d) as in (b)

$$\sum_{n\geq 0} \sum_{k=0}^{n} \binom{n}{k} B_{n-k}\, z^k \frac{x^n}{n!}$$

$$= \sum_{n\geq 0} \sum_{m\geq 0} B_m \frac{x^m}{m!} \frac{(zx)^n}{n!}$$

$$= \mathcal{B}(x)\, e^{zx} = \mathcal{B}(x,z)$$

(e) and

$$\sum_{k=0}^{n-1} k^{m-1} = \frac{B_m(n) - B_m}{m}$$

$$\Leftrightarrow m \sum_{m\geq 1} \left(\sum_{k=0}^{n-1} k^{m-1}\right) \frac{x^m}{m!} = \sum_{m\geq 1} B_m(n) \frac{x^m}{m!} - \sum_{m\geq 1} B_m \frac{x^m}{m!}$$

$$\Leftrightarrow x \sum_{m\geq 1} \sum_{k=0}^{n-1} \frac{(xk)^{m-1}}{(m-1)!} = \left(\frac{x e^{nx}}{e^x-1} - 1\right) - \left(\frac{x}{e^x-1} - 1\right)$$

$$\iff \quad x \sum_{k=0}^{n-1} \sum_{m\geq 1} \frac{(xk)^{m-1}}{(m-1)!} = \frac{x\,(e^{nx}-1)}{e^{x}-1}$$

$$\iff \quad x \sum_{k=0}^{n-1} e^{kx} = x \left(\frac{e^{nx}-1}{e^{x}-1} \right)$$

which is clear

③ (a) $$\sin(z) = z \prod_{n\geq 1} \left(1 - \frac{z^2}{n^2\pi^2}\right)$$

so $$\frac{\cos(z)}{\sin(z)} = \frac{z'}{z} + \sum_{n\geq 1} \frac{\left(1-\frac{z^2}{n^2\pi^2}\right)'}{1-\frac{z^2}{n^2\pi^2}}$$

$$= \frac{1}{z} + 2\sum_{n\geq 1} \frac{-\frac{z}{n^2\pi^2}}{1-\frac{z^2}{n^2\pi^2}} = \frac{1}{z} + 2\sum_{n\geq 1} \frac{z}{z^2-n^2\pi^2}$$

so $$z \operatorname{cotan}(z) = 1 + 2 \sum_{n\geq 1} \frac{z^2}{z^2-n^2\pi^2}$$

(b) $$\frac{x}{e^x-1} = B(x) = \sum B_n \frac{x^n}{n!}$$

$$B(2iz) = \frac{2iz}{e^{2iz}-1}$$

$$\operatorname{cotan}(z) = \frac{\cos(z)}{\sin(z)} = i \cdot \frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}$$

$$= i\,\frac{e^{2iz}+1}{e^{2iz}-1} = i\left(1 + \frac{2}{e^{2iz}-1}\right)$$

so $$B(2iz) = iz\left(\frac{\operatorname{cotan}(z)}{i} - 1\right) = z\operatorname{cotan}(z) - iz$$

$$\parallel$$

$$\sum B_n \frac{(2iz)^n}{n!}$$

so $$z \operatorname{cotan}(z) = iz + 1 + \underbrace{B_1 \frac{2iz}{1}}_{= -iz} + \sum_{n\geq 2} (2i)^n B_n \frac{z^n}{n!}$$

$$= 1 + \sum_{n\geq 2} B_n (2i)^n \frac{z^n}{n!}$$

(c) $$\frac{z^2}{z^2-n^2\pi^2} = 1 + \frac{1}{\left(\frac{z}{n\pi}\right)^2-1}$$

$$= 1 - \sum_{k\geq 0}\left(\frac{z}{n\pi}\right)^{2k} = -\sum_{k\geq 1}\frac{z^{2k}}{n^{2k}\pi^{2k}}$$

so $$z\,\mathrm{cotan}(z) = 1 + 2\sum_{n=1}^{\infty}\frac{z^2}{z^2-n^2\pi^2} = 1 - 2\sum_{\substack{n\geq 1\\ k\geq 1}}\frac{z^{2k}}{n^{2k}\pi^{2k}}$$

$$= 1 - 2\sum_{k\geq 1}\frac{z^{2k}}{\pi^{2k}}\sum_{n\geq 1}\frac{1}{n^{2k}} = 1 - 2\sum_{k\geq 1}\frac{z^{2k}\zeta(2k)}{\pi^{2k}}$$

(d) $$1 - 2\sum_{k\geq 1}\frac{z^{2k}\zeta(2k)}{\pi^{2k}} = 1 + \sum_{n\geq 2}(2i)^n B_n\frac{z^n}{n!}$$

so $$-\frac{2\zeta(2k)}{\pi^{2k}} = (2i)^{2k}\frac{B_{2k}}{(2k)!}$$

so $$\zeta(2k) = (-1)^{k+1}\frac{B_{2k}\,(2\pi)^{2k}}{2\,(2k)!}$$

(e) $$\Lambda(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$ satisfies $\Lambda(1-s) = \Lambda(s)$

so $$\pi^{-\frac{1-2n}{2}}\Gamma\left(\frac{1-2n}{2}\right)\zeta(1-2n) = \pi^{-n}\Gamma(n).\,\zeta(2n)$$

so $$\zeta(1-2n) = \frac{\pi^{\frac{1}{2}-2n}\,\Gamma(n)\,\zeta(2n)}{\Gamma\left(\frac{1}{2}-n\right)} \quad \frac{2h}{3d}$$

$$= \frac{\pi^{\frac{1}{2}-\cancel{2n}}\,\cancel{(n-1)}!\;(-1)^{n+1}\frac{B_{2n}\,(\cancel{2\pi})^{2n}}{2\,\cancel{(2n)}!}}{(\cancel{-4})^n\, n!\,\frac{\cancel{4}\sqrt{\pi}}{\cancel{(2n)}!}} =$$

$$= -\frac{B_{2n}}{2n} \quad \square.$$

so $\zeta(-1) = -\frac{B_2}{2} = -\frac{1}{12}$

$\quad\ \ \|$

"$1+2+3+\cdots$"

(4) (a). $$z \cot z = 1 + 2\sum_{n\geq 1} \frac{z^2}{z^2 - n^2\pi^2}$$

so $$\pi z \cot(\pi z) = 1 + 2\sum_{n\geq 1} \frac{z^2}{z^2-n^2}$$

so $$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n\geq 1}\left(\frac{2z}{z^2-n^2}\right)$$

$$= \frac{1}{z} + \sum_{n\geq 1}\left(\frac{1}{z+n} + \frac{1}{z-n}\right)$$

(b) $q = e^{2\pi i z}$ $\qquad \cos(\pi z) = \frac{e^{\pi i z} + e^{-\pi i z}}{2} = \frac{\sqrt{q}+\frac{1}{\sqrt{q}}}{2}$

$$\sin(\pi z) = \frac{e^{\pi i z} - e^{-\pi i z}}{2i} = \frac{\sqrt{q}-\frac{1}{\sqrt{q}}}{2i}$$

so $$\cot(\pi z) = i\,\frac{\sqrt{q}+\frac{1}{\sqrt{q}}}{\sqrt{q}-\frac{1}{\sqrt{q}}} = i\,\frac{q+1}{q-1} = i + \frac{2i}{q-1}$$

$$\pi \cot(\pi z) = \pi i - \frac{2\pi i}{1-q} = +\pi i - 2\pi i \sum_{n=0}^{\infty} q^n.$$

(c) $$\frac{\left(\pi \cot \pi z\right)^{(k-1)}}{(k-1)!} = \frac{(-1)^{k-1}}{z^k} + \sum_{n\geq 1}\left(\frac{(-1)^{k-1}}{(z+n)^k} + \frac{(-1)^{k-1}}{(z-n)^k}\right)$$

$$= \sum_{n\in\mathbb{Z}} \frac{(-1)^{k-1}}{(z+n)^k}$$ which converges for $k \geq 2$

also $q' = (e^{2\pi i z})' = 2\pi i\, q$

and $(q^n)' = 2\pi i n\, q^n$

so $(q^n)^{(k-1)} = (2\pi i n)^{k-1} q^n$

so $(\pi \cot \pi z)^{(k-1)} = -2\pi i \sum_{n=1}^{\infty} (2\pi i n)^{k-1} q^n$

so comparing get
$$\sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{n \geq 1} n^{k-1} q^n. \qquad \square$$

(d)
$$G_{2k}(z) = \sum_{\substack{m=0 \\ n \neq 0}} \frac{1}{(mz+n)^{2k}} + \sum_{\substack{m \neq 0 \\ n \in \mathbb{Z}}} \frac{1}{(mz+n)^{2k}}$$

as $(-1)^{2k} = 1$,

$$= 2 \sum_{n \geq 1} \frac{1}{n^{2k}} + 2 \sum_{\substack{m \geq 1 \\ n \in \mathbb{Z}}} \frac{1}{(mz+n)^{2k}}$$

$$= 2\zeta(2k) + 2 \sum_{\substack{m \geq 1 \\ n \in \mathbb{Z}}} \frac{1}{(mz+n)^{2k}}$$

$$= 2\zeta(2k) + 2 \sum_{m \geq 1} \left(\sum_{n \in \mathbb{Z}} \frac{1}{(mz+n)^{2k}} \right)$$

$$= 2\zeta(2k) + 2 \sum_{m \geq 1} \frac{(-2\pi i)^{2k}}{(2k-1)!} \sum_{n \geq 1} n^{2k-1} \underbrace{e^{2\pi i m n z}}_{q^{mn}}$$

$$= 2\zeta(2k) + 2 \sum_{m,n \geq 1} \frac{(-2\pi i)^{2k}}{(2k-1)!} n^{2k-1} q^{mn}$$

rewrite as n

rewrite as d

$$= 2\zeta(2k) + 2 \sum_{n \geq 1} \frac{(-2\pi i)^{2k}}{(2k-1)!} \sum_{d \mid n} d^{2k-1} q^n$$

$$= 2\zeta(2k) + \frac{2(-2\pi i)^{2k}}{(2k-1)!} \sum_{n \geq 1} \sigma_{2k-1}(n) q^n$$

(e) $$E_{2k} = \frac{2(2k-1)!\,\zeta(2k)}{2(-2\pi i)^{2k}} + \sum_{n\geq 1} \sigma_{2k-1}(n) q^n$$

$$\frac{2\,(2k-1)!\,\zeta(2k)}{2\cdot(-2\pi i)^{2k}} = \frac{\cancel{2}\,\cancel{(2k-1)!}}{\cancel{2}\cdot\cancel{(2\pi)}^{\cancel{2k}}\,\cancel{(-1)}^{\cancel{k}}} (-1)^{\cancel{k+1}} \frac{B_{2k}\,\cancel{(2\pi)}^{\cancel{2k}}}{2\,(2k)!}$$

$$= -\frac{B_{2k}}{4k} = \frac{\zeta(1-2k)}{2} \in \mathbb{Q}$$

so $E_{2k} \in \mathbb{Q}[[q]]$

(f) $$E_{12} = -\frac{B_{12}}{24} + \sum_{n\geq 1} \sigma_{11}(n) q^n$$

$$E_{12} = \frac{691}{65520} + \sum_{n\geq 1} \sigma_{11}(n) q^n \equiv \sum_{n\geq 1} \sigma_{11}(n) q^n \pmod{691}$$