HW 7 Solution

③ optional ...

④ (a) $G = G_{\mathbb{Q}(\zeta_p)/\mathbb{Q}} \simeq (\mathbb{Z}/p\mathbb{Z})^\times \simeq \mathbb{Z}/(p-1)\mathbb{Z}$

$g \leftarrow$ generator $\longleftrightarrow 1$

$\chi(g) = \zeta_{p-1}$ $\qquad g^{p-1} = 1$

$\chi^{\frac{p-1}{2}}(g^a) = \chi(g^{\frac{p-1}{2}\cdot a})$ $\qquad (\Rightarrow g^{\frac{p-1}{2}} = -1$

$= \chi((-1)^a)$

$= (-1)^a$ $\qquad$ as χ must be odd eg $\chi(g^{\frac{p-1}{2}}) = \zeta_{p-1}^{\frac{p-1}{2}} = -1$

$= \left(\frac{g^a}{p}\right)$

(b) $[G:H] = 2$ so $[K^H:\mathbb{Q}] = 2$

K^H is quadratic and $\mathrm{Gal}(K^H/\mathbb{Q}) \twoheadleftarrow G_{K/\mathbb{Q}} \simeq \mathbb{Z}/(p-1)\mathbb{Z}$

but $\mathbb{Z}/(p-1)\mathbb{Z}$ has only one quotient $\simeq \mathbb{Z}/2\mathbb{Z}$

namely $\mathbb{Z}/(p-1)\mathbb{Z} \big/ 2\mathbb{Z}/(p-1)\mathbb{Z}$

and so $G_{K^H/\mathbb{Q}} \simeq G_{\mathbb{Q}(\sqrt{p^*})/\mathbb{Q}}$

so $K^H = \mathbb{Q}(\sqrt{p^*})$.

(c) $\chi^{k+\frac{p-1}{2}}(g^{2a}) = \chi^k(g^{2a})\,\chi^{\frac{p-1}{2}}(g^{2a})$

$= \chi^k(g^{2a})\,\chi(g^{p-1}) = \chi^k(g^{2a})$

so $\chi^k|_H = \chi^{k+\frac{p-1}{2}}|_H$.

Thus $G_{\mathbb{Q}(\sqrt{p^*})/\mathbb{Q}} = G/H$

has characters 1 and $\chi^{\frac{p-1}{2}}$ because

$$\{\chi^k\} / \{\chi^k = \chi^{k+\frac{p-1}{2}}\} = \{1, \chi^{\frac{p-1}{2}}\}$$

and from (a) these are 1 and $\left(\frac{\cdot}{p}\right)$.

Finally, from class $\prod_{\chi \in \widehat{G_{\mathbb{Q}(\sqrt{p^*})}}} \tau(\chi) = \begin{cases} \sqrt{p} & p^* = \\ i\sqrt{p} & p^* = - \end{cases}$

$$\tau(1)\,\tau\left(\left(\frac{\cdot}{p}\right)\right)$$

so $\tau\left(\left(\frac{\cdot}{p}\right)\right) = \sqrt{p^*}$

(d) $p^* = -p$

$$\zeta_{\mathbb{Q}(\sqrt{p^*})}(s) = \zeta(s)\, L\left(\left(\frac{\cdot}{p}\right), s\right)$$

so $L\left(\left(\frac{\cdot}{p}\right), 1\right) = \dfrac{2\pi \cdot h_{\mathbb{Q}(\sqrt{p^*})}}{w_{\mathbb{Q}(\sqrt{p^*})}\sqrt{p}}$ ← no regulator as $r_1 = 0$

if $p = 3$ then $w_{\mathbb{Q}(\sqrt{-3})} = 6$ $\quad \{\pm 1, \pm\zeta_3, \pm\zeta_3^2\}$

if $p > 3$, $\equiv 3\ (4)$ $\quad w_{\mathbb{Q}(\sqrt{p^*})} = 2$ $\quad \{\pm 1\}$

so $L\left(\left(\frac{\cdot}{p}\right), 1\right) = \begin{cases} \dfrac{\pi}{3\sqrt{3}} & p = 3 \quad \text{as } h_{\mathbb{Q}(\sqrt{-3})} = 1 \\ \dfrac{\pi\, h_{\mathbb{Q}(\sqrt{p})}}{\sqrt{p}} & p > 3. \end{cases}$

Know $L\left(\left(\frac{\cdot}{p}\right), 1\right) = \dfrac{+\pi i \tau\left(\left(\frac{\cdot}{p}\right)\right)}{p} B_{1,\left(\frac{\cdot}{p}\right)}$

$$= \frac{\pi i \sqrt{-p}}{p} B_{1,\left(\frac{\cdot}{p}\right)}.$$

so $$\frac{\pi\, h_{\mathbb{Q}(\sqrt{-p})}}{\sqrt{p}} = L\left(\left(\frac{\cdot}{p}\right),1\right) = -\frac{\pi}{\sqrt{p}} B_{1,\left(\frac{\cdot}{p}\right)}$$

extra 3 if $p=3$

and so $$B_{1,\left(\frac{\cdot}{p}\right)} = \begin{cases} -h_{\mathbb{Q}(\sqrt{-p})} & p>3 \\ -\frac{1}{3} & p=3. \end{cases}$$

Finally $$B_{1,\psi} = \frac{1}{f_\psi} \sum_{k=1}^{f_\psi} k\,\psi(k)$$

so $$B_{1,\left(\frac{\cdot}{p}\right)} = \frac{1}{p}\sum_{k=1}^{p} \left(\frac{k}{p}\right) k$$ and the result follows

(e) $p \equiv 1 \pmod 4$ $\Rightarrow$ $p^* = p$ $\quad \tau\left(\left(\frac{\cdot}{p}\right)\right) = \sqrt{p}$

$$R_{\mathbb{Q}(\sqrt{p})} = \left|\log |a+b\sqrt{p}|\right|$$

as before get

$$L\left(\left(\frac{\cdot}{p}\right),1\right) = \lim_{s\to 1} \frac{\zeta_{\mathbb{Q}(\sqrt{p})}(s)}{\zeta(s)} \overset{\substack{r_1=2\\ r_2=0}}{=\!=} \frac{2^2 h_{\mathbb{Q}(\sqrt{p})} R_{\mathbb{Q}(\sqrt{p})}}{2\sqrt{p}}$$

$$= \frac{2\, h_{\mathbb{Q}(\sqrt{p})} \left|\log|a+b\sqrt{p}|\right|}{\sqrt{p}}$$

from claim $$-\frac{\overset{\sqrt{p}}{\tau\left(\left(\frac{\cdot}{p}\right)\right)}}{p} \sum_{k=1}^{p} \left(\frac{k}{p}\right) \log|1-\zeta_p^k|$$

so $$h_{\mathbb{Q}(\sqrt{p})} = -\frac{1}{2\left|\log|a+b\sqrt{p}|\right|} \sum_{k=1}^{p} \left(\frac{k}{p}\right) \log|1-\zeta_p^k|$$

② (a) $u \in \mathcal{O}_K^\times \leftrightarrow N(u) = \pm 1$

$u = a + b\sqrt{3} \rightarrow N(u) = a^2 - 3b^2 = -1$

impossible as $\left(\frac{-1}{3}\right) = -1$.

Let $\mathcal{O}_K^\times = \pm u^{\mathbb{Z}}$ $\quad u = a + b\sqrt{3}$

and we may assume $a > 0$ as $a \neq 0$

and if $a < 0$ take $-u$ instead.

If $b < 0$ take $\frac{1}{a+b\sqrt{3}} = \cancel{\pm} a - b\sqrt{3}$

so $u = a + b\sqrt{3}$ has $a, b > 0$.

Suppose $a + b\sqrt{3} \neq 2 + \sqrt{3}$. Then

$2 + \sqrt{3} = \pm (a + b\sqrt{3})^k$ and $a, b > 0$

implies $2 + \sqrt{3} = (a + b\sqrt{3})^k$ $\quad k > 0$.

$$2 + \sqrt{3} = \left(a^k + 3\binom{k}{2} a^{k-2} b^2 + 3^2 \binom{k}{4} a^{k-4} b^4 + \dots\right)$$
$$+ \sqrt{3}\left(\binom{k}{1} a^{k-1} b + 3\binom{k}{3} a^{k-3} b^3 + \dots\right)$$

so $2 = a^k + 3\binom{k}{2} a^{k-2} b^2 + \dots > a^k$

so $a = 1$ or 2

$a = 1$ $\quad a^2 - 3b^2 < 0$ false

$a = 2$ $\quad a^2 - 3b^2 = 1 \rightarrow b = \pm 1$ $\quad b > 0$

so $b = 1$.

(b) $N(xu) = N(x)$ so same true of $\chi = \text{sign}\, N$

as $N(u) = 1$

$I = (x)$ $\quad \chi(I) = \chi(x)$ well-defined as χ

is well-defined up to a unit.

(c) $O_K = \mathbb{Z}[\sqrt{3}]$ so look at $X^2 - 3 \mod p$

mod 2 $= (X-1)(X+1)$ get $(\sqrt{3}+1)^2 = (2)$

$= (X+1)^2$

mod 3 $= X^2$ get $(\sqrt{3})^2 = (3)$

mod $p > 3$ $\quad p = \begin{cases} u \cdot \bar{u} & \left(\frac{3}{p}\right) = 1 \text{ ie } X^2 - 3 \text{ splits mod } p \\ p & \left(\frac{3}{p}\right) = -1 \end{cases}$

if $p = u \cdot \bar{u}$ so $\left(\frac{3}{p}\right) = 1$

$p > 3$ by quadratic reciprocity

$$\left(\frac{p}{3}\right)\left(\frac{3}{p}\right) = (-1)^{\frac{p-1}{2}\frac{3-1}{2}} = (-1)^{\frac{p-1}{2}}$$

so $\left(\frac{p}{3}\right) = \left(\frac{3}{p}\right) \cdot (-1)^{\frac{p-1}{2}}$

p splits $\rightarrow \left(\frac{p}{3}\right) = (-1)^{\frac{p-1}{2}}$

$N(u) = \pm p = \chi(u) \cdot p.$

$\quad\ \ \|$

$a^2 - 3b^2$ so $\left(\frac{N(u)}{3}\right) = \left(\frac{a^2}{3}\right) = 1$

$$\left(\frac{N(u)}{3}\right) = \left(\frac{\pm p}{3}\right) = \left(\frac{\pm 1}{3}\right) \cdot \left(\frac{p}{3}\right) = \pm 1 \left(\frac{p}{3}\right) = \chi(u)\left(\frac{p}{3}\right)$$

so $\chi(u) = \left(\frac{p}{3}\right) = (-1)^{\frac{p-1}{2}}$

(d) $L(\chi, s) = \left(1 - \frac{\chi(1+\sqrt{3})}{2^s}\right)^{-1} \left(1 - \frac{\chi(\sqrt{3})}{3^s}\right)^{-1} \prod_{p>3} \prod_{\beta | p} \left(1 - \frac{\chi(\beta}{\|\beta\|}\right.$

$N(1+\sqrt{3}) = -2 < 0$
$N(\sqrt{3}) = -3 < 0$

$$= \left(1+\frac{1}{2^s}\right)^{-1}\left(1+\frac{1}{3^s}\right)^{-1} \prod_{\substack{p>3 \\ \left(\frac{3}{p}\right)=-1}} \left(1 - \frac{\chi(p)}{p^{2s}}\right)^{-s} \prod_{\substack{p>3 \\ \left(\frac{3}{p}\right)=1 \\ p = u\cdot\bar{u}}} \left(1 - \frac{\chi(u)}{p^s}\right)^{-1}\left(1 - \frac{\chi(\bar{u})}{p^s}\right.$$

$N(p) = p^2 > 0$

$$= \left(1+\frac{1}{2^s}\right)^{-1}\left(1+\frac{1}{3^s}\right)^{-1} \prod_{p>3\ \left(\frac{3}{p}\right)=-1} \left(1 - \frac{1}{p^{2s}}\right)^{-1} \prod_{\substack{p>3 \\ \left(\frac{3}{p}\right)=1}} \left(1 - \frac{\chi(u)}{p}\right)^{-2}$$

$$L\left(\left(\frac{\cdot}{3}\right), s\right) L\left(\left(\frac{\cdot}{4}\right), s\right) = \prod_{p\neq 3} \left(1 - \frac{\left(\frac{p}{3}\right)}{p^s}\right)^{-1} \prod_{p \neq 2} \left(1 - \frac{\left(\frac{p}{4}\right)}{p^s}\right)^{-1}$$

$$= \left(1 - \frac{\left(\frac{2}{3}\right)}{2^s}\right)^{-1} \left(1 - \frac{\left(\frac{3}{4}\right)}{3^s}\right)^{-1} \prod_{p>3} \left(1 - \frac{\left(\frac{p}{3}\right)}{p^s}\right)^{-1}\left(1 - \frac{\left(\frac{p}{4}\right)}{p^s}\right)^{-1}$$

$$= \left(1+\frac{1}{2^s}\right)^{-1}\left(1+\frac{1}{3^s}\right)^{-1} \prod_{\substack{\left(\frac{p}{3}\right)=1 \\ p>3}} \left(1 - \frac{1}{p^s}\right)^{-1}\left(1 - \frac{(-1)^{\frac{p-1}{2}}}{p^s}\right)^{-1}$$

$$\times \prod_{\left(\frac{p}{3}\right)=-1} \left(1 + \frac{1}{p^s}\right)^{-1}\left(1 - \frac{(-1)^{\frac{p-1}{2}}}{p^s}\right)^{-1}$$

from (c) $\left(\frac{3}{p}\right) = \left(\frac{p}{3}\right)\cdot(-1)^{\frac{p-1}{2}}$

if $p = 2$ or 3 the factors involving p are the same in the two products.

Say $\left(\frac{p}{3}\right) = (-1)^{\frac{p-1}{2}} = 1$ Then $\left(\frac{3}{p}\right) = 1$.

In $L(\chi, s)$ the factors are $\left(1 - \frac{1}{p^s}\right)^2$

$L = L\left(\left(\frac{\cdot}{3}\right), s\right) L\left(\left(\frac{\cdot}{4}\right), s\right)$ —— $\left(1 - \frac{1}{p^s}\right)\left(1 - \frac{1}{p^s}\right)$ $\Big\}$ equal

$\left(\frac{p}{3}\right) = 1 \quad (-1)^{\frac{p-1}{2}} = -1 \quad$ so $\left(\frac{3}{p}\right) = -1$

$L(\chi, s)$ have $1 - \frac{1}{p^{2s}}$

L $\left(1 - \frac{1}{p^s}\right)\left(1 + \frac{1}{p^s}\right)$ $\Big\}$ equal

similarly when $\left(\frac{p}{3}\right) = -1 \quad (-1)^{\frac{p-1}{2}} = \pm 1.$