Introduction to Algebraic Number Theory
Lecture 10

Andrei Jorza

2014-02-05

Today: Norms of ideals, ramification index and inertia index. Textbook here is `http://wstein.org/books/ant/ant.pdf`

5 Ideals under extensions (continued)

(5.3) Finished the proposition from last time.

(5.4) Main theorem about ramification and inertia indices. Recall from last time that if R is a Dedekind domain and $\mathfrak{p}$ is a prime ideal then $|R/\mathfrak{p}^n| = |k_\mathfrak{p}|^n$.

Lemma 1. *Let L/K be number fields and $\mathfrak{p}$ a prime ideal of $\mathcal{O}_K$. Then $\dim_{k_\mathfrak{p}}(\mathcal{O}_L/\mathfrak{p}\mathcal{O}_L) \le [L:K]$.*

Proof. (The proof from class had a couple of issues, fixed here, but which don't affect the argument at all.) Let $n = [L:K]$. We need to show that any $n+1$ elements $\alpha_1, \dots, \alpha_{n+1}$ of $\mathcal{O}_L/\mathfrak{p}\mathcal{O}_L$ have a nontrivial $k_\mathfrak{p}$ dependence. Since $\dim_K L = n$, there exist $\beta_1, \dots, \beta_{n+1} \in K$, not all 0, such that $\sum \alpha_i\beta_i = 0$. Multiplying by suitable integers we may assume that $\beta_i \in \mathcal{O}_K$ and we'd like to find such a dependence such that the images of $\beta_i \in \mathcal{O}_K$ in $k_\mathfrak{p} = \mathcal{O}_K/\mathfrak{p}$ are not all 0. Suppose $\beta_i \in \mathfrak{p}$ for all i. Then the ideal $J = (\alpha_1, \dots, \beta_{n+1}) = \sum \mathcal{O}_K\beta_i \subset \mathfrak{p}$. Let $J^{-1} = \sum \mathcal{O}_K\gamma_i$. Then $JJ^{-1} = \sum \mathcal{O}_K\beta_i\gamma_j = \mathcal{O}_K$ and thus $\beta_i\gamma_j \in \mathcal{O}_K$ for all i,j and $\beta_{i_0}\gamma_{j_0} \notin \mathfrak{p}$ for some i_0, j_0. Then $\sum \alpha_i\beta_i = 0$ implies $\sum \alpha_i\beta_i\gamma_{j_0} = 0$ is a linear dependence among the α_i, with coefficients in $\mathcal{O}_K$ and such that at least one of the coefficients $(\beta_{i_0}\gamma_{j_0})$ does not vanish in $k_\mathfrak{p}$. Thus α_i are dependent over $k_\mathfrak{p}$ and the conclusion follows. $\square$

Theorem 2. *Suppose L/K are number fields.*

1. *If $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_K$ and $\mathfrak{q}_i$ are the distinct prime factors of $\mathfrak{p}\mathcal{O}_L$ then*
$$\sum e_{\mathfrak{q}_i/\mathfrak{p}} f_{\mathfrak{q}_i/\mathfrak{p}} = [L:K]$$
where recall that $\mathfrak{p}\mathcal{O}_L = \prod \mathfrak{q}_i^{e_{\mathfrak{q}_i/\mathfrak{p}}}$ and $f_{\mathfrak{q}_i/\mathfrak{p}} = [k_{\mathfrak{q}_i} : k_\mathfrak{p}]$.

2. *If I is a fractional ideal of K then $||I\mathcal{O}_L|| = ||I||^{[L:K]}$.*

Proof. Note that the norm is multiplicative (from last time) and so
$$||\mathfrak{p}\mathcal{O}_L|| = \prod ||\mathfrak{q}_i||^{e_i} = \prod |k_{\mathfrak{q}_i}|^{e_i} = \prod |k_\mathfrak{p}|^{e_i f_{\mathfrak{q}_i/\mathfrak{p}}} = ||\mathfrak{p}||^{\sum e_i f_i}$$

We first prove (1) for $K = \mathbb{Q}$. Indeed, then $\mathfrak{p} = (p)$ and so $\mathcal{O}_L/p\mathcal{O}_L \cong \mathbb{F}_p^{[L:K]}$ since $\mathcal{O}_L$ is a rank n free $\mathbb{Z}$-module which implies that $p^n = p^{\sum e_i f_i}$ and the conclusion follows.

Next we prove (2). By multiplicativity of the norm of an ideal and the fact that $||aI|| = |N_{K/\mathbb{Q}}(a)|||I||$ it suffices to treat the case of prime ideals $I = \mathfrak{p}$ in which case we need to show that $||\mathfrak{p}\mathcal{O}_L|| = ||\mathfrak{p}||^n$ where $n = [L:K]$. Let p be the prime of $\mathbb{Z}$ below $\mathfrak{p}$ of $\mathcal{O}_K$ and let $(p)\mathcal{O}_K = \prod \mathfrak{p}_i^{e_{\mathfrak{p}_i/\mathfrak{p}}}$. From the lemma we know that $\dim_{k_{\mathfrak{p}_i}} \mathcal{O}_L/\mathfrak{p}_i\mathcal{O}_L \leq [L:K]$ while from part (1) we know that $\sum e_{\mathfrak{p}_i/p} f_{\mathfrak{p}_i/p} = [K:\mathbb{Q}]$ and, equivalently for $L/\mathbb{Q}$, $||(p)\mathcal{O}_L|| = p^{[L:\mathbb{Q}]}$. So

$$\begin{aligned} p^{[L:\mathbb{Q}]} &= ||(p)\mathcal{O}_L|| \\ &= \prod ||\mathfrak{p}_i\mathcal{O}_L||^{e_{\mathfrak{p}_i/p}} \\ &= \prod |\mathcal{O}_L/\mathfrak{p}_i\mathcal{O}_L|^{e_{\mathfrak{p}_i/p}} \\ &\leq \prod |k_{\mathfrak{p}_i}|^{[L:K]e_{\mathfrak{p}_i/p}} \\ &= \prod |\mathbb{F}_p|^{[L:K]f_{\mathfrak{p}_i/p}e_{\mathfrak{p}_i/p}} \\ &= p^{[L:K][K:\mathbb{Q}]} = p^{[L:\mathbb{Q}]} \end{aligned}$$

Therefore all inequalities are equality and so $||\mathfrak{p}_i\mathcal{O}_L|| = ||\mathfrak{p}_i||^{[L:K]}$ for all i and in particular for $\mathfrak{p} = \mathfrak{p}_i$ for some i.

Finally, from (2) we deduce (1). We already know that

$$||\mathfrak{p}\mathcal{O}_L|| = ||\mathfrak{p}||^{\sum e_{\mathfrak{q}_i/\mathfrak{p}} f_{\mathfrak{q}_i/\mathfrak{p}}}$$

and $||\mathfrak{p}\mathcal{O}_L|| = ||\mathfrak{p}||^{[L:K]}$ from part (2) and the conclusion follows.

□