Introduction to Algebraic Number Theory
Lecture 21

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9 Counting Ideals

(9.3)

Proof of Theorem. (Continued from last time).

We need to estimate $n_C(t) = w^{-1}|\iota(J) \cap \mathcal{D}'_{\log(t||J||)}|$.

It is a general analytical statement from the geometry of number that if $\mathcal{C}$ is a region in $\mathbb{R}^n$ with a "nice" boundary $\partial\mathcal{C}$ and $\Lambda \subset \mathbb{R}^n$ is a lattice then

$$\begin{aligned}|\Lambda \cap x\mathcal{C}| &= \frac{\operatorname{vol}(x\mathcal{C})}{\operatorname{vol}(\Lambda)} + O\left(\frac{\operatorname{vol}(\partial x\mathcal{C})}{\operatorname{vol}(\partial\Lambda)}\right) \\ &= x^n \frac{\operatorname{vol}(\mathcal{C})}{\operatorname{vol}(\Lambda)} + O(x^{n-1})\end{aligned}$$

(where $\operatorname{vol}(\partial\Lambda)$ represents the surface area of the fundamental parallelotope).

We will apply this to $\Lambda = \iota(J)$ and $\mathcal{C} = \mathcal{D}'_{\log(||J||)}$. First, note that

$$\mathcal{D}'_{\log(t||J||)} = \sqrt[n]{t}\mathcal{D}'_{\log(||J||)}$$

Indeed, under the log map the region $x\mathcal{D}'_{\log(||J||)}$ becomes $(\log(x), \dots, \log(x), 2\log(x), \dots, 2\log(x)) + \mathcal{D}_{\log(||J||)}$ which by definition is just $\mathcal{D}_{n\log(x)+\log(||J||)} = \mathcal{D}_{\log(x^n||J||)}$.

Thus

$$\begin{aligned}n_C(t) &= w^{-1}|\iota(J) \cap \mathcal{D}'_{\log(t||J||)}| \\ &= w^{-1}|\iota(J) \cap \sqrt[n]{t}\mathcal{D}'_{\log(||J||)}| \\ &= w^{-1}\frac{\operatorname{vol}(\mathcal{D}'_{\log(||J||)})}{\operatorname{vol}(\iota(J))}t + O(t^{1-1/n})\end{aligned}$$

and we only need to show that

$$\kappa = \frac{\operatorname{vol}(\mathcal{D}'_{\log(||J||)})}{w\operatorname{vol}(\iota(J))} = \frac{2^r(2\pi)^s R_K}{w\sqrt{|\operatorname{disc}(K)|}}$$

We will only compute $\operatorname{vol}(\mathcal{D}'_{\log(||J||)})$ for real quadratic fields since the general statement has no new ideas and a different language (that of adeles) is more suitable for the general computation.

Let $K = \mathbb{Q}(\sqrt{m})$ with $m > 0$ square-free and not 1. Then $r = 2, s = 0, w = 2$ (the units must be real) and if u is a fundamental unit for $\mathcal{O}_K^\times$ then $R_K = |\log|u||$ for any of the two real embeddings of u into $\mathbb{R}$. Then $\mathcal{F} = \{x(|\log|u||, -|\log|u||)|x \in [0,1)\} = \{x(R_K, -R_K)|x \in [0,1)\}$. Thus $\mathcal{D}$ consists of $\{(x,y)\}$ with $y \le x$, $y \ge x - 2R_K$ (look at the graph) and $\mathcal{D}_{\log(||J||)}$ has the extra condition that $x + y \le \log(||J||)$.

What is $\mathcal{D}'_{\log(||J||)}$? In the first quadrant it is the region bounded by $y = x$, $y = xe^{-2R_K}$ and $xy = ||J||$, but because $\ker\log = \{\pm1\}^2$, it is mirror images of this region in the other three quadrants. Thus (this is single variable calculus)

$$\begin{aligned}\mathrm{vol}(\mathcal{D}'_{\log(||J||)} &= 4\,\mathrm{vol}(\text{bounded by } y=x, y=xe^{-2R_K}, xy=||J||) \\ &= 4\int_{\sqrt{||J||}}^{\sqrt{||J||e^{2R_K}}} \frac{||J||}{x}dx \\ &= 4R_K||J||\end{aligned}$$

We've computed before that $\mathrm{vol}(\iota(J)) = 2^{-s}||J||\sqrt{|\operatorname{disc}(K)|}$ (in the proof of the Minkowski bound). Putting everything together we get

$$\begin{aligned}\kappa &= \frac{\mathrm{vol}(\mathcal{D}'_{||J||})}{w\,\mathrm{vol}(\iota(J))} \\ &= \frac{2^2||J||R_K}{w||J||\sqrt{|\operatorname{disc}(K)|}} \\ &= \frac{2^2R_K}{2\sqrt{|\operatorname{disc}(K)|}}\end{aligned}$$

□

10 ζ-functions and L-functions

(10.1)

Definition 1. Suppose $(a_n)_{n\geq 1}$ is a sequence of complex numbers. The **Dirichlet series** of (a_n) is

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

Example 2. The Riemann ζ-function

$$\zeta(s) = 1 + \frac{1}{2^s} + \cdots$$

is the Dirichlet series of $a_n = 1$.

Lemma 3. *If $A_t = \sum_{n=1}^{t} a_n = O(t^r)$ for some real number r then the Dirichlet series $\sum \frac{a_n}{n^s}$ converges on $\mathrm{Re}(s) > r$ and is holomorphic in that region.*

Proof. If $|A_t| \leq Bt^r$ for some B then

$$\begin{aligned}|\sum_{n=1}^{t}\frac{a_n}{n^s}| &= |\sum_{n=1}^{t}\frac{A_n - A_{n-1}}{n^s}| \\ &= |\frac{A_t}{t^s} - A_1 + \sum_{n=1}^{t-1} A_n\left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right)| \\ &\leq Bt^{r-\mathrm{Re}\,s} + |A_1| + B\sum_{n=1}^{t-1} n^r\left|\frac{1}{n^s} - \frac{1}{(n+1)^s}\right| \\ &\leq Bt^{r-\mathrm{Re}\,s} + |A_1| + B|s|\sum_{n=1}^{t-1} n^{r-\mathrm{Re}\,s-1} \\ &\leq Bt^{r-\mathrm{Re}\,s} + |A_1| + B|s| + B|s|\left(\frac{(t-1)^{r-\mathrm{Re}\,s}-1}{r-\mathrm{Re}\,s}\right)dx\end{aligned}$$

and this converges when $\operatorname{Re} s > r$ as desired. Holomorphicity follows from the fact that this convergence is uniform on compact sets. $\square$