Introduction to Algebraic Number Theory
Lecture 22

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10 ζ-functions and L-functions

(1.1) Let K be a number field.

Definition 1. The Dedekind ζ-function is

$$\zeta_K(s) = \sum_{I \subset \mathcal{O}_K} \frac{1}{||I||^s}$$

Proposition 2. *$\zeta_K(s)$ converges and is holomorphic for* $\mathrm{Re}(s) > 1$.

Proof.

$$\begin{aligned}\zeta_K(s) &= \sum_I \frac{1}{||I||^s} \\ &= \sum_{t=1}^{\infty} \sum_{||I||=t} \frac{1}{t^s}\end{aligned}$$

Writing a_n for the number of ideals of norm n it follows that $n_K(t) = \sum_{n=1}^t a_n = O(t)$ and convergence follows from the lemma. ☐

(1.2) "Analytic continuation"

Theorem 3 (Analytic Class Number Formula)**.** *Let K be a number field.*

1. *The Riemann ζ-function $\zeta(s)$ can be extended to a meromorphic function on* $\mathrm{Re}\, s > 0$ *with a simple pole at $s = 1$ and*
$$\lim_{s \to 1} (s-1)\zeta(s) = 1$$
2. *The Dedekind ζ-function $\zeta_K(s)$ can be extended to a meromorphic function on* $\mathrm{Re}\, s > 1 - 1/[K : \mathbb{Q}]$ *with a simple pole at $s = 1$ with*
$$\lim_{s \to 1} (s-1)\zeta_K(s) = \frac{2^r (2\pi)^s h_K R_K}{w\sqrt{|\,\mathrm{disc}(K)|}}$$

Proof. Part one. The function $f(s) = (1 - 2^{1-s})\zeta(s)$ can be written as

$$f(s) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s}$$

for $\mathrm{Re}\, s > 1$ but the latter is is holomorphic for $\mathrm{Re}\, s > 0$ by the lemma as $\sum_{n=1}^t (-1)^{n-1} = O(1)$. This implies that $\zeta(s)$ is meromorphic with poles possibly when $2^{1-s} = 1$, i.e., when $(1-s)\log(2) = 2\pi i k$ for some $k \in \mathbb{Z}$.

Similarly the function $g(s) = (1 - 3^{1-s})\zeta(s)$ can be written as

$$g(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

for $\operatorname{Re} s > 1$ where $a_n = 1$ unless $3 \mid n$ in which case $n = -2$. Again $g(s)$ makes sense as a holomorphic function when $\operatorname{Re} s > 0$ and so $\zeta(s)$ is meromorphic with poles possibly when $3^{1-s} = 1$, i.e., when $(1 - s)\log(3) = 2\pi i \ell$ for some $\ell \in \mathbb{Z}$.

Suppose $\zeta(s)$ has a pole at some s such that $(1-s)\log(2) = 2\pi i k$ and $(1-s)\log(3) = 2\pi i \ell$. Then $2^\ell = 3^k$ and so $\ell = k = 0$ and $s = 1$. Thus $\zeta(s)$ is meromorphic with only possible pole at $s = 1$. Let's compute the residue:

$$\begin{aligned} \lim_{s \to 1}(s-1)\zeta(s) &= \lim_{s \to 1} \frac{f(s)(s-1)}{1 - 2^{1-s}} \\ &= \frac{f(1)}{\log(2)} \\ &= 1 \end{aligned}$$

as

$$f(1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log(1+1) = \log(2)$$

Part two. Recall that for $\operatorname{Re} s > 1$ one has

$$\begin{aligned} \zeta_K(s) &= \sum_{n=1}^{\infty} \frac{n_K(n) - n_K(n-1)}{n^s} \\ &= h_K \kappa \zeta(s) + \sum_{n=1}^{\infty} \frac{n_K(n) - n_K(n-1) - \kappa h_K}{n^s} \end{aligned}$$

Again by our lemma it follows that $\zeta_K(s) - h_K\kappa\zeta(s)$ is holomorphic for $\operatorname{Re}(s) > 1 - 1/[K:\mathbb{Q}]$ since

$$\sum_{n=1}^{t}(n_K(n) - n_K(n-1) - \kappa h_K) = n_K(t) - \kappa h_K t = O(t^{1-1/[K:\mathbb{Q}]})$$

This implies that $\zeta_K(s) - h_K\kappa\zeta(s)$ is holomorphic for $\operatorname{Re} s > 1 - 1/[K:\mathbb{Q}]$ and so the same must be true of $\zeta_K(s)$. For the residue computation note that

$$\begin{aligned} \lim_{s \to 1}(s-1)\zeta_K(s) &= \lim_{s \to 1}(s-1)(\zeta_K(s) - h_K\kappa\zeta(s)) + h_K\kappa \lim_{s \to 1}(s-1)\zeta(s) \\ &= h_K\kappa \end{aligned}$$

as in the first limit one has the product of two functions which are continuous at $s = 1$.

□

(1.3) Functional equation.

Recall the Euler Γ function:

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx$$

We will use two variants:

$$\begin{aligned} \Gamma_{\mathbb{R}}(s) &= \pi^{-s/2}\Gamma\left(\frac{s}{2}\right) \\ \Gamma_{\mathbb{C}}(s) &= 2(2\pi)^{-s}\Gamma(s) \end{aligned}$$

Lemma 4. *1.* $\Gamma(x+1) = x\Gamma(x)$

2. $\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = 2^{1-2x}\sqrt{\pi}\Gamma(2x)$ *and* $x=\frac{1}{2}$ *gives* $\Gamma(1/2)=\sqrt{\pi}$.

3. $\Gamma(n) = (n-1)!$ *for* $n \geq 1$.

Proof. Not given. □

Theorem 5. *Let K be a number field with r_1 real and $2r_2$ complex places. Write $d_K = |\operatorname{disc}(K)|$ and*

$$\Lambda(s) = d_K^{s/2}\Gamma_{\mathbb{R}}(s)^{r_1}\,\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_K(s)$$

Then $\Lambda(s) = \Lambda(1-s)$.

Proof. Not given. Proof is better given in a different language. □

Corollary 6 (A basic version of Birch and Swinnerton-Dyer)**.** *The function ζ_K has a zero of order r_1+r_2-1 at $s=0$ and*

$$\frac{\zeta_K^{(r_1+r_2-1)}(0)}{(r_1+r_2-1)!} = \frac{h_K R_K}{w}$$

Here the order of vanishing r_1+r_2-1 is the rank of the finitely generated abelian group $\mathcal{O}_K^\times = \operatorname{Res}_{K/\mathbb{Q}}\mathbb{G}_m(\mathbb{Z})$.

Proof. Next time. □