Introduction to Algebraic Number Theory
Lecture 27

Andrei Jorza

11 Special values of the ζ-function and of L-functions

(11.5) Gauss sums.

Definition 1. Suppose χ is a character. The Gauss sum

$$\tau(\chi) = \sum_{a=1}^{f_\chi} \chi(a) e^{2\pi i a/f_\chi}$$

For example if $\chi_3 = \left(\frac{\cdot}{3}\right)$ then $\tau(\chi_3) = \zeta_3 - \zeta_3^2 = i\sqrt{3}$.

Lemma 2. *Suppose $b \in \mathbb{Z}$ and χ is a character of conductor f. Then*

$$\sum_{a=1}^{f} \overline{\chi}(a) e^{2\pi i ab/f} = \chi(b)\tau(\overline{\chi})$$

and so $\overline{\tau(\chi)} = \chi(-1)\tau(\overline{\chi})$.

Proof. If $(b, f) = 1$ then $\{ab | a \in \mathbb{Z}/f\mathbb{Z}\} = \mathbb{Z}/f\mathbb{Z}$ and so

$$\begin{aligned}
\chi(b)\tau(\overline{\chi}) &= \chi(b) \sum_{a=1}^{f} \overline{\chi}(a) e^{2\pi i a/f} \\
&= \chi(b) \sum_{a=1}^{f} \overline{\chi}(ab) e^{2\pi i ab/f} \\
&= \chi(b)\overline{\chi}(b) \sum_{a=1}^{f} \overline{\chi}(a) e^{2\pi i ab/f} \\
&= \sum_{a=1}^{f} \overline{\chi}(a) e^{2\pi i ab/f}
\end{aligned}$$

since $\chi(b)\overline{\chi}(b) = |\chi(b)| = 1$ because $\operatorname{Im}\chi$ consists of roots of unity.

If $(b, f) = d > 1$ then the RHS vanishes as $\chi(b) = 0$. Write $b = cd$ and $f = gd$. The character $\overline{\chi}$ has conductor f and so it does not come from a character modulo g. In other words $\overline{\chi}$ is not trivial on fibers of the quotient $(\mathbb{Z}/f\mathbb{Z})^\times \twoheadrightarrow (\mathbb{Z}/g\mathbb{Z})^\times$. For the fiber over 1 this means that for some $u \equiv 1 \pmod{g}$ (in the fiber over 1) and coprime to f the character $\overline{\chi}(u) \neq 1$. But then multiplying by u coprime to f permutes terms so

$$\begin{aligned}
\sum_{a=1}^{f} \overline{\chi}(a) e^{2\pi i ab/f} &= \sum_{a=1}^{f} \overline{\chi}(au) e^{2\pi i abu/f} \\
&= \overline{\chi}(u) \sum_{a=1}^{f} \overline{\chi}(a) e^{2\pi i ab/f}
\end{aligned}$$

and so the LHS is also 0 as $\overline{\chi}(u) \neq 1$.

For the last statement, apply with $b = -1$ and note that $e^{-2\pi i a/f}$ is the conjugate of $e^{2\pi i a/f}$. $\square$

Lemma 3. $|\tau(\chi)| = \sqrt{f_\chi}$.

Proof. Since $|\chi(b)| = 1$ if $(b, f) = 1$ and 0 otherwise we get (using the previous lemma)

$$\begin{aligned}
\varphi(f)|\tau(\chi)|^2 &= \sum_{b=1}^{f} |\chi(b)\tau(\chi)|^2 \\
&= \sum_{b=1}^{f} \overline{\chi(b)\overline{\tau(\chi)}}\chi(b)\overline{\tau(\chi)} \\
&= \sum_{b=1}^{f} \overline{\chi(b)\tau(\overline{\chi})}\chi(b)\tau(\overline{\chi}) \\
&= \sum_{b=1}^{f}\sum_{a=1}^{f} \chi(a)e^{-2\pi i ab/f} \sum_{c=1}^{f} \overline{\chi}(c)e^{2\pi i cb/f} \\
&= \sum_{a=1}^{f}\sum_{c=1}^{f} \chi(a)\overline{\chi}(c) \sum_{b=1}^{f} e^{2\pi i(c-a)b/f}
\end{aligned}$$

But the RHS is either 0 if $a \neq c$ or f if $a = c$ and so

$$\begin{aligned}
\varphi(f)|\tau(\chi)|^2 &= f\sum_{a=1}^{f} |\chi(a)|^2 \\
&= f\varphi(f)
\end{aligned}$$

which gives the desired result. $\square$

(11.6) The functional equation. A section containing two theorems without proofs because either they are too hard or unilluminating.

Definition 4. A character χ is said to be odd if $\chi(-1) = -1$. It is even if $\chi(-1) = 1$.

Theorem 5. *Suppose χ is a character of conductor f_χ. If $\chi(-1) = -1$ let $\delta_\chi = 1$ and if $\chi(-1) = 1$ let $\delta_\chi = 0$. Then*

$$f_\chi^{s/2}\Gamma_{\mathbb{R}}(s + \delta_\chi)L(\chi, s) = W_\chi f_\chi^{(1-s)/2}\Gamma_{\mathbb{R}}(1 - s + \delta_\chi)L(\overline{\chi}, 1 - s)$$

where $W_\chi = \dfrac{\tau(\chi)}{i^{\delta_\chi}\sqrt{f_\chi}}$.

Recall that we showed in class that if $K = \mathbb{Q}(\zeta_N)$ then

$$\zeta_K(s) = \prod_{\mathfrak{p}|N}\left(1 - \frac{1}{||\mathfrak{p}||^s}\right)\prod_{\chi \mod N} L(\chi \mod N, s)$$

Theorem 6. *If $K/\mathbb{Q}$ is abelian then*

$$\zeta_K(s) = \prod_{\chi} L(\chi, s)$$

where χ ranges through the character of the abelian Galois group $\mathrm{Gal}(K/\mathbb{Q})$.

(11.7) The value at 1.

Theorem 7. *Suppose χ is a nontrivial character.*

1. *If $\chi(-1) = -1$ (χ is said to be odd) then*

$$L(\chi, 1) = \frac{\pi i \tau(\chi)}{f_\chi} B_{1,\overline{\chi}}$$

2. *If $\chi(-1) = 1$ (χ is said to be even) then*

$$L(\chi, 1) = -\frac{\tau(\chi)}{f_\chi} \sum_{a=1}^{f_\chi} \overline{\chi}(a) \log|1 - \zeta_{f_\chi}^a|$$

Proof. Part one: Using the functional equation for χ odd with $\delta_\chi = 1$ we get

$$\begin{aligned} L(\chi, 1) &= \frac{W_\chi f_\chi^{-1/2} \Gamma_{\mathbb{R}}(1) L(\overline{\chi}, 0)}{\Gamma_{\mathbb{R}}(2)} \\ &= -\frac{\pi \tau(\chi) B_{1,\overline{\chi}}}{i f_\chi} \\ &= \frac{\pi i \tau(\chi)}{f_\chi} B_{1,\overline{\chi}} \end{aligned}$$

where $\Gamma_{\mathbb{R}}(2) = \pi^{-1}\Gamma(2) = \pi^{-1}$ and $\Gamma_{\mathbb{R}}(1) = \pi^{-1/2}\Gamma(1/2) = 1$.

Part two: For $\chi(-1) = 1$ and $\chi \neq 1$ everything converges in the following computation. We are using Lemma 2 for replacing $\chi(n)$ with Gauss sums.

$$\begin{aligned} L(\chi, 1) &= \sum_{n \geq 1} \frac{\chi(n)}{n} \\ &= \sum_{n \geq 1} \frac{1}{n \tau(\overline{\chi})} \sum_{a=1}^{f} \overline{\chi}(a) e^{2\pi i a n / f} \\ &= \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{f} \overline{\chi}(a) \sum_{n \geq 1} \frac{1}{n} e^{2\pi i a n / f} \\ &= -\frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{f} \overline{\chi}(a) \log(1 - \zeta_f^a) \end{aligned}$$

But $\tau(\overline{\chi}) = \chi(-1)\overline{\tau(\chi)} = \overline{\tau(\chi)} = f/\tau(\chi)$ and $\log(1 - \zeta_f^a) + \log(1 - \zeta_f^{-a}) = 2\log|1 - \zeta_f^a|$ and so

$$\begin{aligned} L(\chi, 1) &= -\frac{\tau(\chi)}{f} \frac{1}{2} \sum_{a=1}^{f} (\overline{\chi}(a) \log(1 - \zeta_f^a) + \overline{\chi}(-a) \log(1 - \zeta_f^{-a})) \\ &= -\frac{\tau(\chi)}{f} \sum_{a=1}^{f} \overline{\chi}(a) \log|1 - \zeta_{f_\chi}^a| \end{aligned}$$

since $\chi(-1) = 1$.

□

Corollary 8. *If χ is odd then $B_{1,\chi} \neq 0$.*

Proof. Follows from the previous theorem and the fact that $L(\chi, 1) \neq 0$. There is no elementary proof of this. □

Example 9. If $\chi_3 = \left(\frac{\cdot}{3}\right)$ then we compute $B_{1,\overline{\chi_3}} = -1/3$ and we already computed $\tau(\chi_3) = i\sqrt{3}$ and $f_\chi = 3$ and so we deduce that

$$L(\chi_3, 1) = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \cdots = \frac{\pi}{3\sqrt{3}}$$