Introduction to Algebraic Number Theory
Lecture 28

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11 Special values of the ζ-function and of L-functions

(11.8) The conductor-discriminant formula.

Lemma 1. *Let K be a number field which is Galois over $\mathbb{Q}$ with abelian Galois group G. Recall that K is then either totally real ($r_1 = n = [K : \mathbb{Q}], r_2 = 0$) or totally complex ($r_1 = 0, r_2 = n/2$).*

1. *If K is totally real then every character $\chi \in \widehat{G}$ is even.*

2. *If K is totally complex then $n/2$ of the n characters $\chi \in \widehat{G}$ are even and $n/2$ are odd.*

Proof. By Kronecker-Weber $K \subset \mathbb{Q}(\zeta_N)$ for some N and then $G \cong G_{\mathbb{Q}(\zeta_N)/\mathbb{Q}}/G_{\mathbb{Q}(\zeta_N)/K}$. Let $\sigma \in G_{\mathbb{Q}(\zeta_N)/\mathbb{Q}}$ correspond to $-1 \in (\mathbb{Z}/N\mathbb{Z})^\times \cong G_{\mathbb{Q}(\zeta_N)/\mathbb{Q}}$. The automorphism σ takes ζ_N to ζ_N^{-1} and so $\zeta(z) = \overline{z}$ is simply complex conjugation.

Also denote by σ is image in G and let $\psi \in \widehat{\widehat{G}}$ be associated to σ, i.e., $\psi(chi) = \chi(\sigma)$. Here σ is the image of -1 and so $\psi(\chi) = \chi(-1)$ which is either 1 or -1. Thus to count the even/odd characters it suffices to count $\ker \psi$ in which case $|\ker \psi|$ is the number of even characters. But $|\ker \psi| = n/|\operatorname{Im} \psi|$ and this is either n if $\operatorname{Im} \psi = 1$ or $n/2$ if $\operatorname{Im} \psi = \{-1, 1\}$.

If $\sigma \neq 1$ in G then there exists a character χ such that $\chi(\sigma) \neq 1$ (for example the identity character on the quotient $G \to \langle \sigma \rangle \cong \{-1, 1\}$) and so $\operatorname{Im} \psi = 1$ if and only if $\sigma \neq 1$. But σ is complex conjugation and this is trivial in G if and only if it fixed K if and only if K is totally real. $\square$

Theorem 2 (Conductor-discriminant)**.** *Let K be a number field with abelian Galois group G over $\mathbb{Q}$.*

1. $\prod_{\chi \in \widehat{G}} f_\chi = |\operatorname{disc}(K)|$.

2. $\prod_{\chi \in \widehat{G}} \tau(\chi) = \begin{cases} \sqrt{|\operatorname{disc}(K)|} & K \textit{ totally real} \\ i^{[K:\mathbb{Q}]/2}\sqrt{|\operatorname{disc}(K)|} & K \textit{ totally complex} \end{cases}.$

Proof. Write $d_K = |\operatorname{disc}(K)|$. Recall the functional equations

$$\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_K(s) = d_K^{1/2-s}\Gamma_{\mathbb{R}}(1-s)^{r_1}\Gamma_{\mathbb{C}}(1-s)^{r_2}\zeta_K(1-s)$$

and

$$\Gamma_{\mathbb{R}}(s+\delta_\chi)L(\chi, s) = W_\chi f_\chi^{1/2-s}\Gamma_{\mathbb{R}}(1-s+\delta_\chi)L(\overline{\chi}, 1-s)$$

and the decomposition

$$\zeta_K(s) = \prod_{\chi \in G} L(\chi, s)$$

Totally real case: $r_1 = n, r_2 = 0$ and by the lemma all characters are even and so all $\delta_\chi = 0$. Divide the first functional equation by the product of the second ones over $\chi \in \widehat{G}$. Get

$$\frac{\Gamma_{\mathbb{R}}(s)^n \zeta_K(s)}{\prod_{\chi \in \widehat{G}} \Gamma_{\mathbb{R}}(s) L(\chi, s)} = \frac{d_K^{1/2-s} \Gamma_{\mathbb{R}}(1-s)^n \zeta_K(1-s)}{\prod_\chi W_\chi f_\chi^{1/2-s} \Gamma_{\mathbb{R}}(1-s) L(\overline{\chi}, 1-s)}$$
$$1 = \frac{1}{\prod W_\chi} \left(\frac{d_K}{\prod f_\chi} \right)^{1/2-s}$$
$$\prod W_\chi = \left(\frac{d_K}{\prod f_\chi} \right)^{1/2-s}$$

Since the LHS is a constant it follows that $d_K / \prod f_\chi$ must be 1 or else its powers are not constant. This implies the first part. For the second part (using $\delta_\chi = 0$):

$$\begin{aligned} 1 &= \prod W_\chi \\ &= \prod \frac{\tau(\chi)}{i^{\delta_\chi} \sqrt{f_\chi}} \\ &= \frac{\prod \tau(\chi)}{i^{n/2} \sqrt{\prod f_\chi}} \end{aligned}$$

which implies that $\prod \tau(\chi) = i^{n/2} \sqrt{d_K}$.

For K totally complex: $r_1 = 0, r_2 = n/2$ and $n/2$ of the δ_χ are 0 and $n/2$ are 1. As above we get

$$\frac{\Gamma_{\mathbb{C}}(s)^{n/2} \zeta_K(s)}{\prod_{\text{half}} \Gamma_{\mathbb{R}}(s) \prod_{\text{half}} \Gamma_{\mathbb{R}}(s+1) \prod L(\chi, s)} = \frac{d_K^{1/2-s} \Gamma_{\mathbb{C}}(1-s)^{n/2} \zeta_K(1-s)}{\prod_\chi W_\chi f_\chi^{1/2-s} \prod_{\text{half}} \Gamma_{\mathbb{R}}(1-s) \prod_{\text{half}} \Gamma_{\mathbb{R}}(1-s+1) \prod L(\overline{\chi}, 1-s)}$$
$$\left(\frac{\Gamma_{\mathbb{C}}(s)}{\Gamma_{\mathbb{R}}(s) \Gamma_{\mathbb{R}}(s+1)} \right)^{n/2} \frac{\zeta_K(s)}{\prod L(\chi, s)} = \frac{d_K^{1/2-s}}{\prod_\chi W_\chi f_\chi^{1/2-s}} \left(\frac{\Gamma_{\mathbb{C}}(1-s)}{\Gamma_{\mathbb{R}}(1-s) \Gamma_{\mathbb{R}}(1-s+1)} \right)^{n/2} \frac{\zeta_K(1-s)}{\prod L(\overline{\chi}, 1-s)}$$

The proof from the totally real case goes through after noticing that

$$\begin{aligned} \Gamma_{\mathbb{R}}(s) \Gamma_{\mathbb{R}}(s+1) &= \pi^{-1/2-s} \Gamma(s/2) \Gamma(s/2 + 1/2) \\ &= 2^{1-s} \pi^{-s} \Gamma(2s) \\ &= \Gamma_{\mathbb{C}}(s) \end{aligned}$$

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