ALGEBRAIC NUMBER THEORY
LECTURE 33

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Recall 0
Let X be a projective variety. Let $I \subset K[x_0, \cdots, x_n]$ be a prime ideal generated by a homogeneous ideal. Then $K[X] = k[x_0, \cdots, x_n]/I$ is an integral domain, with $K(X)$ it's fraction field, and $\dim(X) = \text{trdeg}K(X)/K$, and recall that X is smooth at P if the Jacobian has rank equal to $\dim X$. Think of a point of X over an algebraic extension as a $\text{Gal}(\bar{K}/L)$ orbit of points over $\bar{K}$. Last time we saw that for a curve C, if P is smooth then $K(C)_p = \{\frac{f}{g} \in K(C),\, g(P) \neq 0\}$ is a division ring. $\mathfrak{m}_P = \{\frac{f}{g} \in K(C)_P |\, f(P) = 0\}$ is its unique maximal ideal, and all ideal are of the form m_P^n for $n > 0$. For $f \in K(C)_p$ $ord_P(f)$ is the largest n such that $f \in m_p^n$. f is a *uniformizer* if $ord_P(f) = 1$. Think of $ord_P(f)$ as the order of the zero at P if n is positive, and the order of the pole is if n is negative.

Ramification

Let $\psi : C_1 \to C_2$ be a morphism between twp smooth projective curves C_1, C_2. Recall this induces a map $\psi* : K(C_2 \to C_1)$ and $deg\psi = [K(C_1) : \psi^* K(C_2)]$.
Definition 1 Let $Q = \psi(P)$ and define $e_{P/Q,\psi} = ord_P(\psi^* \mathfrak{t}_Q)$, where $\mathfrak{t}_Q$ is the uniformizer.
Example 2 Let

$$\psi : \mathbb{P}^1 \to \mathbb{P}^1$$
$$(x : y) \to (x^3(x-y)^2 : y^5)$$

Here deg $\psi = 5$. Let little x $x = \frac{x}{y}$, $y \neq 0$. Then $\psi(x) = x^3(x-1)^2$, and with $\infty = (1 : 0)$, $\psi(\infty) = \infty$. The uniformizer at $Q = \lambda$ is $x - \lambda = t_Q$. $\psi^*(t_Q) = \psi(x) - \lambda = x^3(x-1)^2 - \lambda$, with order of vanishing 2. We have that $P_\lambda(x) = x^3(x-1)^2 - \lambda$. $P_\lambda^{'}(x) = x^2(x-1)(5x-3)$. They are both zero if either i. $x = 0$ and $\lambda = 0$, and $e_{0|0,\psi} = 3$, ii. $x = 1$ $\lambda = 0$, $e_{1|0,\psi} = 2$, iii. $x = 3/5, \lambda = \frac{3^2 2^2}{5^5}, e_{3|5,\psi} = 2$. $e_{\infty|\infty} = 5$. These are all P/Q where $e_{P/Q,\psi} > 1$, i.e. P/Q ramifies.
Proposition 3
a. $\displaystyle\sum_{P \in \psi^{-1}} e_{P/Q,\psi} = \text{deg } \psi$.
b. Almost all P/Q are unramified.
c. $e_{P/R} = e_{P/Q} e_{P/R}$.
Suppose a field K of characteristic p is perfect, i.e. $\phi(x) = x^p$ is surjective onto K. This gives a map of curves: $\phi : C \to C^{(p)}$, where C = vanishing of $I = (f_1(x_i), \cdots, f_n(x_i)$. and $C^{(p)}$ = vanishing of $I^{(p)} = (f_i^{(p)})$, $f = \sum a_I x^I$, $f^{(p)} =$

$\sum a_I \phi(x^I)$. (eg) C = the line $x + 26 = 3z$ in $\mathbb{P}^2$. Then $C^{(p)}$ = the line $x^p + 2y^p = 3z^p \subset \mathbb{P}^2$ and $\phi(x_0 : \cdots : x_n) = (x_0^p, \cdots, x_n^p)$. If $f(x) = 0$, then it is easy to check that $f^{(p)}(\phi(x)) = 0$. $f^p(\phi(x)) = \sum a_I^p x^{pI} = (\sum a_I x^I)^p = 0$. so $\phi : C \to C^{(p)}$ is a morphism.

Proposition 4

(a.) ϕ has deg p

(b.) ϕ is purely inseparable, ie. $K(C)/K(C^{(p)})$ is purely inseparable of deg p.

Proposition 5

(1) If $a \in K(C) - K$, then $K(C)/K(a)$ is finite.

(2) If $a \notin K(C^{(p)})$, then $K(C)/K(a)$ is separable.

(3) If $\mathfrak{t}$ = uniformizer at a smooth point $K(C)/K(\mathfrak{t})$ is finite separable.

Proof (1) Use what we learned from last time for $K(C)/K(\mathfrak{t})$, t uniformizer.

a. Since it is a finite extension, $K(a, \mathfrak{t})/K(\mathfrak{t})$ is finite. There exists a $A_k(t) \in K(\mathfrak{t})$ such that $\sum A_k(t)a^k = 0$. Reshuffle, and get that $\mathfrak{t}$ satisfies a polynomial $K(a)$. so $k(t, a)/k(a)$ is finite.

Theorem 6

(1) $\text{Frac}(\mathcal{O}_{K(C),S}) = K(C)$

(2) $\mathcal{O}_{K(C),S}$ = Dedekind Domain

$\mathcal{O}_{K(C),S}$ may just be K. e.g. $\mathcal{O}_{K(\mathbb{P}^1),\phi} = K$ finite field.

Application 7

$P(x), Q(x) \in \mathbb{F}_q[x]$ co-prime polynomial irreducible $f \cong P mod Q$.

$$\delta(f(x) \in \mathbb{F}_q[x]) = \frac{1}{\#(\mathbb{F}_q[x]/Q(x))^x}$$