Math 40520 Theory of Number
Homework 1

Due Wednesday, 2015-09-09, in class

Do 5 of the following 7 problems. Please only attempt 5 because I will only grade 5.

1. Find all rational numbers x and y satisfying the equation $x^2 + y^2 = 5$. [Hint: Use the change of variables $u = x - 2y$ and $v = y - 2x$ and find an equation relating u and v.]

2. Find all rational numbers x and y satisfying the equation $x^2 + 2xy + 3y^2 = 2$. [Hint: Use the change of variables $u = x + y$ and $v = y$ and find an equation relating u and v. Then mimick how we found all Pythagorean triples.]

3. Consider the diophantine equation
$$3x + 5y + 7z = 2$$
   (a) Find a solution with $x, y, z \in \mathbb{Z}$. [Hint: Use the Euclidean algorithm from class.]
   (b) Show that if $3X + 5Y + 7Z = 0$ for some integers X, Y, Z then 3 must divide $Z - Y$.
   (c) Find all integral solutions to the equation.

4. Consider the diophantine equation
$$xy = zt$$
with $x, y, z, t \in \mathbb{Z}$. Show that there exist integers a, b, c, d such that $x = ab$, $y = cd$, $z = ac$, $t = bd$. [Hint: Factor x, y, z, t into primes.]

5. Show that all the solutions to the diophantine equation
$$x^2 + y^2 = z^2 + t^2$$
are of the form
$$x = \frac{mn + pq}{2} \qquad y = \frac{mp - nq}{2}$$
$$z = \frac{mp + nq}{2} \qquad t = \frac{mn - pq}{2}$$
for integers m, n, p, q such that the above formulae yield integers. [Hint: Use the previous exercise.]

6. In this exercise you will solve the equation
$$x^2 + y^2 + z^2 = 1$$
with $x, y, z \in \mathbb{Q}$.
   (a) Let (a, b) be the point of intersection of the (xy)-plane with the line through (x, y, z) and $(0, 0, 1)$. Show that
$$\frac{x}{a} = \frac{y}{b} = 1 - z$$

(b) Show, mimicking the procedure from the Pythagorean triples case, that every rational solution of the diophantine equation is of the form

$$x = \frac{2a}{1+a^2+b^2} \qquad y = \frac{2b}{1+a^2+b^2} \qquad z = \frac{a^2+b^2-1}{1+a^2+b^2}$$

for rationals a, b.

7. Suppose two of the integers $a_1, a_2, \ldots, a_n$ are coprime. Suppose $x_1 = u_1, \ldots, x_n = u_n$ is an integral solution to the diophantine equation

$$a_1x_1 + \cdots + a_nx_n = b$$

Find all the other solutions. [Hint: Cf. exercise 3.]