Math 40520 Theory of Number
Homework 5

Due Wednesday, 2015-10-07, in class

Do 5 of the following 8 problems. Please only attempt 5 because I will only grade 5.

1. Compute $\binom{194871}{1610}$ mod 385. [Hint: Use our theorem for binomial coefficients modulo primes (Lucas' theorem) and the Chinese Remainder Theorem.]

2. (This is a more sophisticated looking, yet easier, version of the previous problem.) A *Sophie-Germaine* prime is a prime p such that $q = 2p+1$ is also a prime (conjecturally there are infinitely many such primes, the largest known having about 200k digits). Suppose $p \geq 7$ is a Sophie-Germaine prime and $q = 2p+1$. Show that
$$\binom{pq + pq^2}{pq} \equiv 30q - 2p \equiv 58p + 30 \pmod{pq}$$
[Hint: Same as for the previous problem, but it's easier to write down the digits in bases p and q.]

3. Compute
$$12^{34^{56^{78}}} \mod 90$$
[Hint: It is much easier to use Euler's theorem in conjunction with the Chinese Remainder Theorem.] (The author of this problem was very proud of having used each digit exactly once. This idiosyncrasy actually makes the problem easier.)

4. Let n be a number such that $n+1$ is divisible by 24. If $d \mid n$ show that 24 divides $d^2 - 1$.

5. Exercise 4.19 on page 82.

6. Exercise 5.8 on page 90.

7. Exercise 5.9 on page 90.

8. Exercise 5.21 on page 96.