Math 40520 Theory of Number
Homework 9

Due Wednesday, 2015-12-02, in class

Do 5 of the following 8 problems. Please only attempt 5 because I will only grade 5.

1. Let p be a prime and $k, n \geq 1$ integers. Show that

$$v_p((p^k n)!) = \frac{n(p^k - 1)}{p - 1} + v_p(n!)$$

2. Let p be a prime.

   (a) For an integer n write $n = pq + r$ where $0 \leq r \leq p - 1$. Show that

$$\prod_{1 \leq d \leq n, (d,p)=1} d \equiv (-1)^q r! \pmod{p}$$

   [Hint: Wilson's theorem.]

   (b) Write $n = \overline{n_d \dots n_1 n_0}_{(p)}$ and $\ell = v_p(n!)$. Conclude that

$$\frac{n!}{p^\ell} \equiv (-1)^\ell n_0! n_1! \cdots n_d! \pmod{p}$$

3. Let p be a prime and m, n two integers. Write $m = \overline{m_d \dots m_1 m_0}_{(p)}$, $n = \overline{n_d \dots n_1 n_0}_{(p)}$ and $m - n = \overline{k_d \dots k_1 k_0}_{(p)}$. Show that if $\ell = v_p\left(\binom{m}{n}\right)$ then

$$p^{-\ell} \binom{m}{n} \equiv (-1)^\ell \prod_{i=0}^{d} \frac{m_i!}{n_i! k_i!} \pmod{p}$$

4. (Variant of Exercise 8.3 on page 146) For a positive integer n and a complex number s define

$$\sigma_s(n) = \sum_{d|n} d^s$$

   (a) Show that if m and n are coprime then $\sigma_s(mn) = \sigma_s(m)\sigma_s(n)$.

   (b) Show that if $n = p_1^{k_1} \cdots p_r^{k_r}$ and $s \neq 0$ then

$$\sigma_s(n) = \prod_{i=1}^{r} \frac{p_i^{s(k_i+1)} - 1}{p_i^s - 1}$$

5. Let $p \equiv 1 \pmod{3}$ be a prime.

   (a) Show that there exists $u \in \mathbb{Z}$ such that $u^2 + u + 1 \equiv 0 \pmod{p}$.

(b) Show that there exist integers x, y such that $p = x^2 + xy + y^2$. [Hint: What is the area of ellipse $x^2 + xy + y^2 = R^2$?]

6. Exercise 8.24 on page 163.

7. Exercise 8.21 on page 163.

8. For a positive integer n let $\tau(n)$ be the number of positive divisors of n. Show that

$$D_{\tau^2}(s) = \frac{\zeta(s)^4}{\zeta(2s)}$$

[Hint: Use the fact from class that τ, and therefore also τ^2, is multiplicative, and then some calculus.]