# Math 40520 Theory of Number
# Homework 10

Due Wednesday, 2015-12-09, in class

**Do 5 of the following 8 problems. Please only attempt 5 because I will only grade 5.**

1. For a positive integer $n$ let $f(n)\#\{(x, y, z, t) \in \mathbb{Z}^4 \mid n = xyzt\}$ the number of ways to write $n$ as an ordered product of 4 integers. For example 12 can be written in 4 ways as $12 \cdot 1 \cdot 1 \cdot 1$, in 12 ways as $6 \cdot 2 \cdot 1 \cdot 1$, in 12 ways as $4 \cdot 3 \cdot 1 \cdot 1$ and 12 ways as $3 \cdot 2 \cdot 2 \cdot 1$ for a total of $f(12) = 40$.

   (a) Show that $D_f(s) = \zeta(s)^4$.

   (b) Show that
   $$f(n) = \sum_{d^2|n} \tau(n/d^2)^2$$
   (For example $f(12) = 40 = \tau(12)^2 + \tau(3)^2 = 6^2 + 2^2$.) [Hint: Compare the Dirichlet series of $\tau^2$ and $f$.]

2. Show that $\mathbb{Z}[\sqrt{3}]$ is a Euclidean domain. [Hint: similar to $\mathbb{Z}[\sqrt{2}]$, but needs one more step.]

3. Consider the Euclidean domain $R = \mathbb{Z}[i]$. Find the gcd of $x = 21 + 47i$ and $y = 62 + 9i$ using the Euclidean algorithm.

4. Consider the Euclidean domain $R = \mathbb{Z}[\sqrt{2}]$ and let $x = 36 - 19\sqrt{2}$ and $y = 35 - 31\sqrt{2}$. Compute the Bézout identity: find the gcd $d = (x, y)$ and two elements $p, q \in \mathbb{Z}[\sqrt{2}]$ such that $d = xp + yq$.

5. Show that $2, 3, 1 \pm \sqrt{-5}$ are irreducible in the domain $\mathbb{Z}[\sqrt{-5}]$, but they are not prime. Conclude that $\mathbb{Z}[\sqrt{-5}]$ is not a Euclidean domain.

6. Show that if a prime integer $p$ is $\equiv \pm 3 \pmod 8$ then $p$ is a prime element of the domain $\mathbb{Z}[\sqrt{2}]$.

7. Consider the set $\mathbb{Z}[\sqrt[3]{2}] = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} \mid a, b, c \in \mathbb{Z}\}$ as a subset of $\mathbb{R}$.

   (a) Show that $\mathbb{Z}[\sqrt[3]{2}]$ is a domain. [Hint: check if it is closed under $+, -, \cdot$.]

   (b) Take for granted that $N(a + b\sqrt[3]{2} + c\sqrt[3]{4}) = a^3 + 2b^3 + 4c^3 - 8abc$ satisfies the following two properties: i. $N(xy) = N(x)N(y)$ for all $x, y$ of the form $a + b\sqrt[3]{2} + c\sqrt[3]{4}$ and ii. $N(x) = 0$ if and only if $x = 0$ (this is a boring exercise). If $a, b, c \in (0, 1/2)$ show that $N(a+b\sqrt[3]{2}+c\sqrt[3]{4}) \in (-1, 1)$.

   (c) Show that $\mathbb{Z}[\sqrt[3]{2}]$ is a Euclidean domain with Euclidean function $d(x) = |N(x)|$.

   ($\mathbb{Z}[\sqrt[3]{3}]$ also has $d(x) = |N(x)|$ as a Euclidean function but these are the only two examples of this kind.)