Graduate Algebra
Homework 1

Due 2015-01-28

1. Let R be a commutative ring. An *enhanced* R-module is a pair (M, f) of an R-module M and an endomorphism $f \in \mathrm{End}_R(M)$. A homomorphism of enhanced R-modules $\phi : (M, f) \to (N, g)$ is an R-module homomorphism $\phi : M \to N$ such that $\phi \circ f = g \circ \phi$. Define $(M, f) \oplus (N, g) = (M \oplus N, f \oplus g)$, $(M, f) \otimes_R (N, g) = (M \otimes_R N, f \otimes g)$, $\mathrm{Sym}^k(M, f) = (\mathrm{Sym}^k M, \mathrm{Sym}^k f)$ and $\wedge^k(M, f) = (\wedge^k M, \wedge^k f)$.
   (a) Let (M, f) and (N, g) as above. Show that $\wedge^k(M \oplus N, f \oplus g) \cong \bigoplus_{i+j=k} \wedge^i(M, f) \otimes_R \wedge^j(N, g)$.
   (b) (Optional) The analogous statement for Sym.

2. Let $V = \mathbb{C}^2$ and $f \in \mathrm{End}_{\mathbb{C}}(V)$ given by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{C})$. For an explicit basis of $\mathrm{Sym}^2(V)$ of your choice find the matrix representing $\mathrm{Sym}^2(f)$.

3. Let $A_1, \ldots, A_m \in M_{n\times n}(\mathbb{C})$ be pair-wise commuting matrices, not all 0.
   (a) Show that the subring $R = \mathbb{C}[A_1, \ldots, A_m]$ of $M_{n\times n}(\mathbb{C})$ is $R \cong \mathbb{C}[X_1, \ldots, X_m]/I$ for some proper ideal $I \subset \mathbb{C}[X_1, \ldots, X_m]$.
   (b) Suppose $\mathfrak{m} = (X_1 - \lambda_1, \ldots, X_m - \lambda_m)$ is a maximal ideal containing I. If $Q(X_1, \ldots, X_m)$ is any polynomial, show that $Q(\lambda_1, \ldots, \lambda_m)$ is an eigenvalue of $Q(A_1, \ldots, A_m)$. (You may assume the so-called weak nullstellensatz which states that every maximal ideal of $\mathbb{C}[X_1, \ldots, X_m]$ is of the form $(X_1 - \lambda_1, \ldots, X_m - \lambda_m)$; you showed this for $m = 2$ last semester.)

4. Let $A \in M_{2\times 2}(\mathbb{C})$.
   (a) Let $f(x) \in \mathbb{C}[[X]]$ be an absolutely converging power series. Show that $f(A)$ converges to an element of $M_{2\times 2}(\mathbb{C})$. [The topology here is that of $\mathbb{C}^4$.]
   (b) Show that $\det(e^A) = e^{\mathrm{Tr}\, A}$.
   (c) Show that $\sin^2(A) + \cos^2(A) = I_2$.
   (d) (Optional) Conclude that if $\cos(A)$ is upper triangular with 1 on the diagonal then $\cos(A) = I_2$.

5. Let $A \in M_{n\times n}(F)$. Writing $V = F^n$ as a module over $F[X]$ via $P(X) \cdot v := P(A)v$ we showed in class that $V \cong F[X]/(P_1(X)) \oplus \cdots \oplus F[X]/(P_k(X))$ for polynomials $P_i(X) \in F[X]$.
   (a) Show that the characteristic polynomial of the matrix
   $$\begin{pmatrix} 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & \ldots & 0 & -a_1 \\ 0 & 1 & \ldots & 0 & -a_2 \\ & & \ddots & & \vdots \\ & & & 1 & -a_{d-1} \end{pmatrix}$$
   is $X^d + a_{d-1}X^{d-1} + \cdots + a_1 X + a_0$.
   (b) Deduce that $P_A(X) = P_1(X) \cdots P_k(X)$.