Graduate Algebra
Homework 1

Due 2015-01-28

1. Let R be a commutative ring. An *enhanced* R-module is a pair (M, f) of an R-module M and an endomorphism $f \in \mathrm{End}_R(M)$. A homomorphism of enhanced R-modules $\phi : (M, f) \to (N, g)$ is an R-module homomorphism $\phi : M \to N$ such that $\phi \circ f = g \circ \phi$. Define $(M, f) \oplus (N, g) = (M \oplus N, f \oplus g)$, $(M, f) \otimes_R (N, g) = (M \otimes_R N, f \otimes g)$, $\mathrm{Sym}^k(M, f) = (\mathrm{Sym}^k M, \mathrm{Sym}^k f)$ and $\wedge^k(M, f) = (\wedge^k M, \wedge^k f)$.

   (a) Let (M, f) and (N, g) as above. Show that

   $$\wedge^k(M \oplus N, f \oplus g) \cong \bigoplus_{i+j=k} \wedge^i(M, f) \otimes_R \wedge^j(N, g)$$

   (b) (Optional) The analogous statement for Sym.

   Proof. Let (m_i) be a basis for M of rank r and (n_j) be a basis for N of rank s. Then $(v_i) = (m_i) \cup (n_j)$ is a basis for $M \oplus N$. We know that a basis for $\wedge^k(M \oplus N)$ is formed by expressions $v_{i_1} \wedge \ldots \wedge v_{i_k}$ for $1 \leq i_1 < \ldots < i_k \leq r + s$ where v_{i_j} is either m_i or n_j.

   Suppose this expression has a basis vectors from M and $b = k - a$ basis vectors from N. Then $v_{i_1} \wedge \ldots \wedge v_{i_a} = m_{i_1} \wedge \ldots \wedge m_{i_a}$ and $v_{i_{a+1}} \wedge \ldots \wedge v_{i_k} = n_{j_1} \wedge \ldots \wedge n_{j_b}$ and so we get an isomorphism

   $$\begin{aligned}
   \wedge^k(M \oplus N) &= \langle v_{i_1} \wedge \ldots \wedge v_{i_k} \rangle \\
   &= \langle (m_{i_1} \wedge \ldots \wedge m_{i_a}) \wedge (n_{j_1} \wedge \ldots \wedge n_{j_b}) | a + b = k \rangle \\
   &= \langle (m_{i_1} \wedge \ldots \wedge m_{i_a}) \otimes (n_{j_1} \wedge \ldots \wedge n_{j_b}) | a + b = k \rangle \\
   &= \bigoplus_{a+b=k} \langle m_{i_1} \wedge \ldots \wedge m_{i_a} \rangle \otimes \langle n_{j_1} \wedge \ldots \wedge n_{j_b} \rangle \\
   &= \bigoplus_{a+b=k} \wedge^a M \otimes \wedge^b N
   \end{aligned}$$

   We can replace $\wedge$ by $\otimes$ since there is no ambiguity on the order.

   We only need to keep track of the homomorphism $\wedge^k(f \oplus g)$ under this decomposition. Note, however, that

   $$\begin{aligned}
   \wedge^k(f \oplus g)(m_{i_1} \wedge \ldots \wedge m_{i_a}) \wedge (n_{j_1} \wedge \ldots \wedge n_{j_b}) &= \wedge^a f(m_{i_1} \wedge \ldots \wedge m_{i_a}) \wedge \wedge^b g(n_{j_1} \wedge \ldots \wedge n_{j_b}) \\
   &\in \wedge^a M \otimes_R \wedge^b N
   \end{aligned}$$

   Thus $\wedge^k(f \oplus g)$ invaries each subspace $\wedge^a M \otimes_R \wedge^b N$ and on this piece it acts as $\wedge^a f \otimes \wedge^b g$ as desired. □

2. Let $V = \mathbb{C}^2$ and $f \in \mathrm{End}_{\mathbb{C}}(V)$ given by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{C})$. For an explicit basis of $\mathrm{Sym}^2(V)$ of your choice find the matrix representing $\mathrm{Sym}^2(f)$.

Proof. Write $f(e_1) = ae_1 + be_2$ and $f(e_2) = ce_1 + de_2$. Take e_1^2, e_1e_2, e_2^2 as a basis of $\mathrm{Sym}^2 V$. Then

$$\begin{aligned}\mathrm{Sym}^2 f(e_1^2) &= (f(e_1))^2 = (ae_1 + be_2)^2 = a^2e_1^2 + 2abe_1e_2 + b^2e_2^2\\ \mathrm{Sym}^2 f(e_1e_2) &= f(e_1)f(e_2) = (ae_1 + be_2)(ce_1 + de_2) = ace_1^2 + (ad + bc)e_1e_2 + bde_2^2\\ \mathrm{Sym}^2 f(e_2^2) &= (f(e_2))^2 = (ce_1 + de_2)^2 = c^2e_1^2 + 2cde_1e_2 + d^2e_2^2\end{aligned}$$

and so the matrix is

$$\mathrm{Sym}^2 f = \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}$$

□

3. Let $A_1, \ldots, A_m \in M_{n\times n}(\mathbb{C})$ be pair-wise commuting matrices, not all 0.

   (a) Show that the subring $R = \mathbb{C}[A_1, \ldots, A_m]$ of $M_{n\times n}(\mathbb{C})$ is $R \cong \mathbb{C}[X_1, \ldots, X_m]/I$ for some proper ideal $I \subset \mathbb{C}[X_1, \ldots, X_m]$.

   (b) Suppose $\mathfrak{m} = (X_1 - \lambda_1, \ldots, X_m - \lambda_m)$ is a maximal ideal containing I. If $Q(X_1, \ldots, X_m)$ is any polynomial, show that $Q(\lambda_1, \ldots, \lambda_m)$ is an eigenvalue of $Q(A_1, \ldots, A_m)$. (You may assume the so-called weak nullstellensatz which states that every maximal ideal of $\mathbb{C}[X_1, \ldots, X_m]$ is of the form $(X_1 - \lambda_1, \ldots, X_m - \lambda_m)$; you showed this for $m = 2$ last semester.)

Proof. (a): Consider $\mathbb{C}[X_1, \ldots, X_m] \to M_{n\times n}(\mathbb{C})$ sending X_i to A_i. This yields a ring homomorphism and I is the kernel of this homomorphism.

(b): Let $\pi : \mathbb{C}[A_1, \ldots, A_m] \to \mathbb{C}[X_1, \ldots, X_m]/\mathfrak{m} \cong \mathbb{C}$ be the natural projection map. Let P_Q be the characteristic polynomial of $Q(A_1, \ldots, A_m)$. Then $S = P_Q(Q(X_1, \ldots, X_m)) \in I$ by Cayley-Hamilton. Thus $\pi(S) = \pi(0) = 0$ and so $\pi \in \mathfrak{m}$ which implies that $S(\lambda_1, \ldots, \lambda_m) = 0$. We deduce that $Q(\lambda_1, \ldots, \lambda_m)$ is an eigenvalue, being a root of P_Q.

□

4. Let $A \in M_{2\times 2}(\mathbb{C})$.

   (a) Let $f(x) \in \mathbb{C}[\![X]\!]$ be an absolutely converging power series. Show that $f(A)$ converges to an element of $M_{2\times 2}(\mathbb{C})$. [The topology here is that of $\mathbb{C}^4$.]

   (b) Show that $\det(e^A) = e^{\mathrm{Tr}\, A}$.

   (c) Show that $\sin^2(A) + \cos^2(A) = I_2$.

   (d) (Optional) Conclude that if $\cos(A)$ is upper triangular with 1 on the diagonal then $\cos(A) = I_2$.

Proof. (a): If $P(X)$ is a polynomial then $P(SAS^{-1}) = SP(A)S^{-1}$. Let B be the Jordan canonical form of A with $A = SBS^{-1}$. Either B is diagonal of the form $\begin{pmatrix}\lambda_1 & \\ & \lambda_2\end{pmatrix}$ or $B = \begin{pmatrix}\lambda & 1 \\ & \lambda\end{pmatrix}$.

In the first case note that $P(A) = SP(B)S^{-1} = S\begin{pmatrix}P(\lambda_1) & \\ & P(\lambda_2)\end{pmatrix}S^{-1}$ and so $f(A) = Sf(B)S^{-1} = S\begin{pmatrix}f(\lambda_1) & \\ & f(\lambda_2)\end{pmatrix}$ converges.

In the second case, suppose $f(x) = \sum a_n x^n$. Then $B^n = \begin{pmatrix}\lambda^n & n\lambda^{n-1} \\ & \lambda^n\end{pmatrix}$ and so $P(B) = \begin{pmatrix}P(\lambda) & P'(\lambda) \\ & P(\lambda)\end{pmatrix}$. Since $f(x)$ converges absolutely it follows that $f'(x)$ converges absolutely and so we have $f(A) = Sf(B)S^{-1} = S\begin{pmatrix}f(\lambda) & f'(\lambda) \\ & f(\lambda)\end{pmatrix}$.

(b): Let B be the Jordan form of A. Then $\det e^A = \det Se^BS^{-1} = \det e^B = e^{\lambda_1}e^{\lambda_2} = e^{\lambda_1+\lambda_2} = e^{\operatorname{Tr} A}$ from our explicit formula for the characteristic polynomial of A.

(c): If A is equivalent to B diagonal then $\sin(A) = S\begin{pmatrix}\sin(\lambda_1) & \\ & \sin(\lambda_2)\end{pmatrix}S^{-1}$ and similarly for $\cos(A)$. Thus $\sin^2(A) + \cos^2(A) = SI_2S^{-1} = I_2$. When $A \sim \begin{pmatrix}\lambda & 1\\ & \lambda\end{pmatrix}$ then $\sin(A) = S\begin{pmatrix}\sin(\lambda) & \cos(\lambda)\\ & \sin(\lambda)\end{pmatrix}S^{-1}$ and similarly for $\cos(A)$. Again the formula is immediate to check.

(d): If $\cos(A) = \begin{pmatrix}1 & x\\ & 1\end{pmatrix}$ then $\cos^2(A) = \begin{pmatrix}1 & 2x\\ & 1\end{pmatrix}$ and so $\sin^2(A) = \begin{pmatrix}0 & -2x\\ & 0\end{pmatrix}$. But then $\sin^2(A)$ has eigenvalues $0, 0$ and so $\sin(A)$ has eigenvalues $0, 0$. But then the characteristic polynomial of $\sin(A)$ is X^2 and Cayley-Hamilton implies $\sin^2(A) = 0_2$ so $x = 0$ as desired. □

5. Let $A \in M_{n\times n}(F)$. Writing $V = F^n$ as a module over $F[X]$ via $P(X)\cdot v := P(A)v$ we showed in class that $V \cong F[X]/(P_1(X)) \oplus \cdots \oplus F[X]/(P_k(X))$ for polynomials $P_i(X) \in F[X]$.

   (a) Show that the characteristic polynomial of the matrix

$$\begin{pmatrix} 0 & 0 & \dots & 0 & -a_0\\ 1 & 0 & \dots & 0 & -a_1\\ 0 & 1 & \dots & 0 & -a_2\\ & & \ddots & & \vdots\\ & & & 1 & -a_{d-1}\end{pmatrix}$$

   is $X^d + a_{d-1}X^{d-1} + \cdots + a_1X + a_0$.

   (b) Deduce that $P_A(X) = P_1(X)\cdots P_k(X)$.

Proof. (a): We'll prove by induction. The case $d = 1$ is trivial. We compute

$$\begin{vmatrix} X & 0 & \dots & 0 & a_0\\ -1 & X & \dots & 0 & a_1\\ 0 & -1 & \dots & 0 & a_2\\ & & \ddots & & \vdots\\ & & & -1 & X + a_{d-1}\end{vmatrix} = X\begin{vmatrix} X & \dots & 0 & a_1\\ -1 & \dots & 0 & a_2\\ & \ddots & & \vdots\\ & & -1 & X + a_{d-1}\end{vmatrix} + (-1)^{d-1}a_0\det(-I_{d-1})$$
$$= X(X^{d-1} + a_{d-1}X^{d-2} + \cdots + a_1) + a_0$$
$$= X^d + a_{d-1}X^{d-1} + \cdots + a_1X + a_0$$

(b): We saw that A acting on V is the same as multiplication by X on $\oplus F[X]/(P_i(X))$. Let m_i be multiplication by X on $F[X]/(P_i(X))$. Then $A = \oplus m_i$ and so $P_A(X) = \prod P_{m_i}(X)$. Also from class we know that if $P_i(X) = X^d + a_{d-1}X^{d-1} + \cdots + a_0$ then multiplication by X on $F[X]/(P_i(X))$ has matrix the one given in part (a), which has characteristic polynomial $P_i(X)$. Thus $P_{m_i}(X) = P_i(X)$ and the result follows. □