Graduate Algebra
Homework 2

Due 2015-02-04

1. Let $A, B, C, D \in M_{n\times n}(\mathbb{C})$.

   (a) Show that
   $$\det\begin{pmatrix} A & B \\ & D \end{pmatrix} = \det(A)\det(D)$$

   (b) If $CD = DC$ show that
   $$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BC)$$

   (c) (Optional) Suppose $A_{i,j} \in M_{n\times n}(\mathbb{C})$ for $1 \le i, j \le k$ are pairwise commuting matrices. Show that
   $$\det(A_{i,j}) = \det(\sum_{\sigma\in S_k} \varepsilon(\sigma)\prod_{i=1}^{k} A_{i,\sigma(i)})$$

2. Let V and W be F-vector spaces of dimensions a and b. Let $F \in \mathrm{End}_F(V)$ and $G \in \mathrm{End}_F(W)$.

   (a) Show that
   $$\det(F \otimes G) = \det(F)^b \det(G)^a$$

   (b) If $F = (f_{i,j})$ and $G = (g_{i,j})$ for some bases of V and W, what is $F \otimes G$ for the tensor product basis of $V \otimes_F W$?

3. Let K be a field and v a discrete valuation on K. For $\alpha \in (1, \infty)$ recall that $|x| := \alpha^{-v(x)}$.

   (a) Show that every point in the interior of an open ball in this metric space is a center for the open ball.

   (b) Show that every open ball in the metric space K is closed.

4. (a) Compute the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{1997})$.

   (b) Let $p > 2$ be a prime. Compute the integral closure of $\mathbb{F}_p[t]$ in $\mathbb{F}_p(\sqrt{t+1})$.

5. Let R be a local integral domain which is not a field. Suppose that the maximal ideal $\mathfrak{m}$ is principal and $\cap\mathfrak{m}^n = 0$. Show that R is a discrete valuation ring.