Graduate Algebra
Homework 2

Due 2015-02-04

1. Let $A, B, C, D \in M_{n\times n}(\mathbb{C})$.

   (a) Show that
   $$\det \begin{pmatrix} A & B \\ & D \end{pmatrix} = \det(A)\det(D)$$

   (b) If $CD = DC$ show that
   $$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BC)$$

   (c) (Optional) Suppose $A_{i,j} \in M_{n\times n}(\mathbb{C})$ for $1 \le i, j \le k$ are pairwise commuting matrices. Show that
   $$\det(A_{i,j}) = \det(\sum_{\sigma\in S_k} \varepsilon(\sigma) \prod_{i=1}^{k} A_{i,\sigma(i)})$$

Proof. (a): Consider S such that SAS^{-1} is upper triangular and T such that TDT^{-1} is upper triangular. Let $U = \begin{pmatrix} S & \\ & T \end{pmatrix}$. Then $U \begin{pmatrix} A & B \\ & D \end{pmatrix} U^{-1} = \begin{pmatrix} SAS^{-1} & SBT^{-1} \\ & TDT^{-1} \end{pmatrix}$ is upper triangular. The determinant of an upper triangular matrix is the product of the diagonal elements and so $\det \begin{pmatrix} A & B \\ & D \end{pmatrix} = \det U \begin{pmatrix} A & B \\ & D \end{pmatrix} U^{-1} = \det SAS^{-1} \det TDT^{-1} = \det A \det D$.

(b): Since $CD = DC$ we have
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} D & \\ -C & I_2 \end{pmatrix} = \begin{pmatrix} AD - BC & B \\ & d \end{pmatrix}$$
and so
$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} \det \begin{pmatrix} D & \\ -C & I_2 \end{pmatrix} = \det \begin{pmatrix} AD - BC & B \\ & D \end{pmatrix}$$
which implies
$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} \det(D)\det(I_2) = \det(AB - CD)\det(D)$$
This implies the result when $\det(D) \neq 0$.

When $\det(D) = 0$ note that for $z \in \mathbb{C}$ not an eigenvalue $\det(D - zI_2) \neq 0$. But C still commutes with $D - zI_2$ and so
$$\det \begin{pmatrix} A & B \\ C & D - zI_2 \end{pmatrix} = \det(A(D - zI_2) - BC)$$
Both LHS and RHS are polynomials in z and the equality occurs for infinitely many values of z. Thus LHS and RHS are equal as polynomials. Plugging in $z = 0$ yields the desired result. □

2. Let V and W be F-vector spaces of dimensions a and b. Let $F \in \mathrm{End}_F(V)$ and $G \in \mathrm{End}_F(W)$.

   (a) Show that
   $$\det(F \otimes G) = \det(F)^b \det(G)^a$$

   (b) If $F = (f_{i,j})$ and $G = (g_{i,j})$ for some bases of V and W, what is $F \otimes G$ for the tensor product basis of $V \otimes_F W$?

Proof. (a): Let v_i be a basis of V and w_j a basis of W. Then $v_i \otimes w_j$ is a basis of $V \otimes_F W$. We compute

$$\begin{aligned}
\det(F \otimes G) \wedge_{i,j} v_i \otimes w_j &= \wedge_j(\wedge_i F(v_i) \otimes G(w_j)) \\
&= \wedge_j(\wedge F(v_i)) \otimes G(w_j) \\
&= \wedge_j \det F(\wedge v_i) \otimes G(w_j) \\
&= (\det F)^b \wedge_j (\wedge v_i) \otimes G(w_j) \\
&= (\det F)^b \wedge_i v_i \otimes (\wedge_j G(w_j)) \\
&= (\det F)^b \wedge_i v_i \otimes \det G \wedge_j w_j \\
&= (\det F)^b (\det G)^a \wedge_{i,j} v_i \otimes w_j
\end{aligned}$$

(b): Write $Fv_i = \sum f_{i,j} v_j$ and $Gw_i = \sum g_{i,j} w_j$. Then $(v_i \otimes w_j)$ is a basis of $V \otimes_F W$ and we compute $(F \otimes G)(v_i \otimes w_j) = F(v_i) \otimes G(w_j) = (\sum f_{i,k} v_k) \otimes (\sum g_{j,l} w_l) = \sum_{k,l} f_{i,k} g_{j,l} v_k \otimes w_l$. Thus the matrix of $F \otimes G$ indexed by pairs (i,j) is $(f_{i,k} g_{j,l})_{(i,j),(k,l)}$. □

3. Let K be a field and v a discrete valuation on K. For $\alpha \in (1, \infty)$ recall that $|x| := \alpha^{-v(x)}$.

   (a) Show that every point in the interior of an open ball in this metric space is a center for the open ball.

   (b) Show that every open ball in the metric space K is closed.

Proof. (a): Suppose $B_{x,r}$ is the open ball of radius r centered at x. Suppose $y \in B$. Then $|x - y| < r$. If $z \in B$ then $|z - y| = |z - x + x - y| \leq \max(|z - x|, |x - y|) < r$. We conclude $B_{x,r} \subset B_{y,r}$. Switching x and y we deduce that $B_{x,r} = B_{y,r}$ as desired.

(b): Consider the open ball $B_{x,r}$. It suffices to show that the complement of $B_{x,r}$ is also open. Suppose $z \in K$ such that $|z - x| \geq r$. If $y \in B_{x,r} \cap B_{z,r}$ then the previous part would imply that $B_{x,r} = B_{y,r} = B_{z,r}$ which is not possible as $|z - x| \geq r$. Thus $B_{x,r} \cap B_{z,r} = \emptyset$. We conclude that

$$K - B_{x,r} = \cup_{z \in K - B_{x,r}} B_{z,r}$$

and so is open. □

4. (a) Compute the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{1997})$.

   (b) Let $p > 2$ be a prime. Compute the integral closure of $\mathbb{F}_p[t]$ in $\mathbb{F}_p(\sqrt{t+1})$.

Proof. (a): Write $d = 1997$. Note that $\mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} | a, b \in \mathbb{Q}\}$. Indeed $\mathbb{Q}(\sqrt{d}) = \{P(\sqrt{d})/Q(\sqrt{d}) | P/Q \in \mathbb{Q}(X)\}$. Since $\sqrt{d}^2 \in \mathbb{Q}$ it follows that $P(\sqrt{d}) = m + n\sqrt{d}$ and similarly for $Q(\sqrt{d})$. Finally $1/(m + n\sqrt{d}) = (m - n\sqrt{d})/(m^2 - dn^2)$. Thus every element of $\mathbb{Q}(\sqrt{d})$ is of the form $a + b\sqrt{d}$ with $a, b \in \mathbb{Q}$.

The element $\alpha = a + b\sqrt{d}$ satisfies the equation $P(X) = X^2 - 2aX + a^2 - db^2 = 0$. If $\alpha \in \mathbb{Q}$ is integral then necessarily $\alpha \in \mathbb{Z}$ as $\mathbb{Z}$ is integrally closed. If $\alpha \notin \mathbb{Q}$ then $P(X)$ is irreducible in $\mathbb{Q}[X]$. Suppose α satisfies the monic integral equation $Q(X) = 0$. Then P and Q have $X - \alpha$ as a common factor in

$\mathbb{Q}(\sqrt{d})[X]$ and so they are not coprime. Since P is irreducible it follows that $P \mid Q$ in $\mathbb{Q}[X]$. But then Gauss' lemma would imply that some multiple of P divides Q in $\mathbb{Z}[X]$. Since Q and P are monic this can only happen if this multiple is P itself and so P must be integral. Then $2a \in \mathbb{Z}$ and so $a = m/2$ for some integer m. Also $a^2 - db^2 = m^2/4 - db^2 \in \mathbb{Z}$. But then $d(2b)^2 \in \mathbb{Z}$ and so the only possible denominator of b (since $d = 1997$) is 2 so $b = n/2$ for some integer n. Thus the integral closure consists of $a + b\sqrt{d}$ with $a = m/2$ and $4 \mid m^2 - 1997n^2$. This is the ring $\mathbb{Z}[\frac{1+\sqrt{d}}{2}]$.

(b): As in part (a), $\mathbb{F}_p(\sqrt{t+1}) = \{a + b\sqrt{t+1} | a, b \in \mathbb{F}_p(X)\}$. Again $\mathbb{F}_p[X]$ is a PID and Gauss' lemma is satisfied. The integral closure consists of $a + b\sqrt{t+1}$ such that $2a \in \mathbb{F}_p[t]$ and $a^2 - (t+1)b^2 \in \mathbb{F}_p[t]$. But $p > 2$ so 2 is invertible in $\mathbb{F}_p[t]$ which implies $a \in \mathbb{F}_p[t]$. Thus $(t+1)b^2 \in \mathbb{F}_p[t]$ and unique factorization implies $b \in \mathbb{F}_p[t]$. We conclude that the integral closure is $\mathbb{F}_p[\sqrt{t+1}]$. □

5. Let R be a local integral domain which is not a field. Suppose that the maximal ideal $\mathfrak{m}$ is principal and $\cap \mathfrak{m}^n = 0$. Show that R is a discrete valuation ring.

Proof. Since $\cap \mathfrak{m}^n = 0$ every $x \in R - 0$ is in some $\mathfrak{m}^n$ for some maximal n. Declare $v(x) = n$. Write $\mathfrak{m} = (\alpha)$ in which case $x \in (\alpha^n)$ implies $x/\alpha^n \in R$. Let $v(x) = n$ and $v(y) = m$ with $n \le m$. Since $x/\alpha^n, y/\alpha^m \in R$ it follows that $(x+y)/\alpha^n \in R$ and so $v(x+y) \ge n$ by definition.

We only need to check that $v(xy) = m + n$. Certainly $xy \in (\alpha^{m+n})$. It suffices to show that $xy \notin (\alpha^{m+n+1})$. If this were the case then $(x/\alpha^n)(y/\alpha^m) \in (\alpha)$ and so the product maps to 0 in $R/\mathfrak{m}$. Since this is a field it follows that one of the two factors is 0 in $R/(\alpha)$ which contradicts the choice of n and m.

Finally take $K = \operatorname{Frac} R$ and $v(x/y) = v(x) - v(y)$ extends v to a valuation on K. Also $\mathcal{O}_v = \{x/y | v(x) \ge v(y)\}$ which we need to check is equal to R. It suffices to show that if $v(y) \le v(x)$ then $x/y \in R$. But $v(y/\alpha^{v(y)}) = 0$ and so $y/\alpha^{v(y)} \in R - \mathfrak{m}$ is a unit. Thus $x/y = (x/\alpha^{v(y)})/(y/\alpha^{v(y)})$ is a fraction of elements of R where the denominator is a unit. □