Graduate Algebra
Homework 3

Due 2015-02-11

1. Consider the complex
$$\cdots \to \mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \to \cdots$$
   (a) Show that the complex is exact.
   (b) Show that the identity map on the complex is not null-homotopic.

2. Let R be a ring. Let $\mathbb{Z}[\mathrm{Mod}_R]$ be the free abelian group generated by R-modules; denote by $[M]$ the generator corresponding to $M \in \mathrm{Mod}_R$. Let $G(R)$ be the quotient of $\mathbb{Z}[\mathrm{Mod}_R]$ by the subgroup generated by $[M]-[M']-[M'']$ for any three R-modules in an exact sequence $0 \to M' \to M \to M'' \to 0$.
   (a) If $M \cong N$ are two R-modules show that $[M] = [N]$ in $G(R)$.
   (b) Show that if $0 \to M_1 \to M_2 \to \cdots \to M_k \to 0$ is a complex of R-modules then
   $$\sum_{i=1}^{k}(-1)^i[M_i] = \sum_{i=1}^{k}(-1)^i[H^i(M^\bullet)]$$
   In particular if the complex $M^\bullet$ is exact then
   $$\sum_{i=1}^{k}(-1)^i[M_i] = 0$$
   in $G(R)$.
   (c) A function $\phi : \mathrm{Mod}_R \to A$ (where A is an abelian group) is said to be additive if $\phi(M) = \phi(M') + \phi(M'')$ for exact sequences $0 \to M' \to M \to M'' \to 0$. Show that ϕ extends to a homomorphism of abelian groups $\phi : G(R) \to A$.

3. Let R be a ring. Let $\mathbb{Z}[\mathrm{Proj}_R]$ be the free abelian group generated by isomorphism classes of finitely generated projective R-modules and let $K_0(R)$ be the quotient by the subgroup generated by $[P \oplus Q] - [P] - [Q]$ for any finitely generated projectives P and Q. (Recall from last semester that a short exact sequence where the third term is projective splits as a direct sum.)
   (a) Show that $[P] \cdot [Q] = [P \otimes_R Q]$ extends to a ring multiplication on the abelian group $K_0(R)$ endowing $K_0(R)$ with the structure of an abelian ring.
   (b) Show that K_0 yields a functor from Rings to Rings.
   (c) Show that $K_0(R) \cong \mathbb{Z}$ for any PID R.

   The ring $K_0(R)$ is the easiest example of algebraic K-theory.

4. Let R be a ring. Consider the following commutative diagram of R-module homomorphisms with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

Show that there exists an exact sequence

$$0 \to \ker a \to \ker b \to \ker c \to \operatorname{coker} a \to \operatorname{coker} b \to \operatorname{coker} c \to 0$$

This is known as the snake lemma.

5. (a) Let $\mathcal{C}$ be the category of local Noetherian commutative rings R such that $R/\mathfrak{m}_R \cong \mathbb{Q}$ and morphisms $f : R \to S$ such that $f(\mathfrak{m}_R) = \mathfrak{m}_S$. Let V be an n-dimensional rational vector space and $T \in \operatorname{End}_{\mathbb{Q}}(V)$. By a *deformation* of (V, T) to $R \in \operatorname{Ob}(\mathcal{C})$ we mean a free R-module V_R of rank n and $T_R \in \operatorname{End}_R(V_R)$ such that $(V_R, T_R) \otimes_R (R/\mathfrak{m}_R, 1) \cong (V, T)$. Show that sending $R \in \operatorname{Ob}(\mathcal{C})$ to the set of deformations of (V, T) to R yields a functor $D : \mathcal{C} \to \text{Sets}$.

   (b) Let $\phi : R \to S$ be a homomorphism of commutative rings giving S the structure of an R-algebra. Let M be an S-module. Let $\operatorname{Der}_R(S, M)$ be the set of R-module homomorphisms $d : S \to M$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in S$. Show that $\operatorname{Der}_R(S, -)$ gives a covariant functor from S-modules to R-modules.