Graduate Algebra
Homework 3

Due 2015-02-11

1. Consider the complex
$$\cdots \to \mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \to \cdots$$
   (a) Show that the complex is exact.
   (b) Show that the identity map on the complex is not null-homotopic.

   Proof. (a): This is trivial.

   (b): Suppose id is null homotopic. Then for each position i, $\mathrm{id}_i = d_i \circ s_{i+1} + s_i \circ d_{i-1}$ for maps of R-modules $s_i : (\mathbb{Z}/4\mathbb{Z})_i \to (\mathbb{Z}/4\mathbb{Z})_{i-1}$. Then for any $x \in (\mathbb{Z}/4\mathbb{Z})_i$ we have $x = d_{i-1}s_i(x) + s_{i+1}d_i(x) = 2s_i(x) + s_{i+1}(2x)$ and the RHS is always even. Taking $x = 1$ yields a contradiction. ☐

2. Let R be a ring. Let $\mathbb{Z}[\mathrm{Mod}_R]$ be the free abelian group generated by R-modules; denote by $[M]$ the generator corresponding to $M \in \mathrm{Mod}_R$. Let $G(R)$ be the quotient of $\mathbb{Z}[\mathrm{Mod}_R]$ by the subgroup generated by $[M]-[M']-[M'']$ for any three R-modules in an exact sequence $0 \to M' \to M \to M'' \to 0$.
   (a) If $M \cong N$ are two R-modules show that $[M] = [N]$ in $G(R)$.
   (b) Show that if $0 \to M_1 \to M_2 \to \cdots \to M_k \to 0$ is a complex of R-modules then
$$\sum_{i=1}^{k}(-1)^i[M_i] = \sum_{i=1}^{k}(-1)^i[H^i(M^\bullet)]$$
   In particular if the complex $M^\bullet$ is exact then
$$\sum_{i=1}^{k}(-1)^i[M_i] = 0$$
   in $G(R)$.
   (c) A function $\phi : \mathrm{Mod}_R \to A$ (where A is an abelian group) is said to be additive if $\phi(M) = \phi(M') + \phi(M'')$ for exact sequences $0 \to M' \to M \to M'' \to 0$. Show that ϕ extends to a homomorphism of abelian groups $\phi : G(R) \to A$.

   Proof. (a): Taking the exact sequuence $0 \to 0 \to 0 \to 0 \to 0$ we deduce that $[0] = 0$. Then $0 \to 0 \to M \to N \to 0$ yields $[M] = [N]$.

   (b): Consider $d_i : M_i \to M_{i+1}$ giving the exact sequence $0 \to \ker d_i \to M_i \to \mathrm{Im}\, d_i \to 0$. Then $[M_i] = [\ker d_i] + [\mathrm{Im}\, d_i]$. We compute
$$\begin{aligned}\sum(-1)^i[M_i] &= \sum(-1)^i([\ker d_i] + [\mathrm{Im}\, d_i]) \\ &= \sum(-1)^i([\ker d_i] - [\mathrm{Im}\, d_{i-1}]) \\ &= \sum(-1)^i[\ker d_i/\mathrm{Im}\, d_{i-1}] \\ &= \sum(-1)^i[H^i(M^\bullet)]\end{aligned}$$

If $M^\bullet$ is exact then all $H^i(M^\bullet) = 0$ and since $[0] = 0$ we deduce that $\sum_{i=1}^{k}(-1)^i[M_i] = 0$.

(c): Define $\phi(\sum a_i[M_i]) := \sum a_i\phi(M_i)$ yielding a homomorphism of abelian groups $\mathbb{Z}[\mathrm{Mod}_R] \to A$. Since ϕ is additive it follows that ϕ vanishes on all generators of the submodule by which we quotient $\mathbb{Z}[\mathrm{Mod}_R]$ to define $G(R)$. The first isomorphism theorem for groups then shows that ϕ factors through the quotient $G(R) \to A$. □

3. Let R be a ring. Let $\mathbb{Z}[\mathrm{Proj}_R]$ be the free abelian group generated by isomorphism classes of finitely generated projective R-modules and let $K_0(R)$ be the quotient by the subgroup generated by $[P \oplus Q] - [P] - [Q]$ for any finitely generated projectives P and Q. (Recall from last semester that a short exact sequence where the third term is projective splits as a direct sum.)

   (a) Show that $[P] \cdot [Q] = [P \otimes_R Q]$ gives a well-defined ring multiplication on the abelian group $K_0(R)$ endowing $K_0(R)$ with the structure of an abelian ring.

   (b) Show that K_0 yields a functor from Rings to Rings.

   (c) Show that $K_0(R) \cong \mathbb{Z}$ for any PID R.

   The ring $K_0(R)$ is the easiest example of algebraic K-theory.

   Proof. (a): To make sense of this as a ring multiplication we first need to show that if P and Q are finitely generated projective then so is $P \otimes_R Q$. The latter is certainly finitely generated so we only need to show that it is projective. Let M and N be such that $P \oplus M = F$ and $Q \oplus N = F'$ are free. Then $F \otimes_R F'$ is free and $F \otimes_R F' = P \otimes_R Q \oplus (P \otimes_R N \oplus M \otimes_R Q \oplus M \otimes_R N)$. Thus $P \otimes_R Q$ is a direct summand of a free module and so it is projective.

   Define multiplication on $\mathbb{Z}[\mathrm{Proj}_R]$ by $(\sum a_i[M_i]) \cdot (\sum b_j[N_j]) = \sum a_i b_j[M_i \otimes_R N_j]$. Then $\mathbb{Z}[\mathrm{Proj}_R]$ is a ring with unit $[R]$. Let I be the subgroup of $\mathbb{Z}[\mathrm{Proj}_R]$ generated by $[P \oplus Q] - [P] - [Q]$. To show that $K_0(R)$ is a ring it suffices to show that I is in fact an ideal of $\mathbb{Z}[\mathrm{Proj}_R]$. For this note that $([P \oplus Q] - [P] - [Q]) \cdot [S] = [P \otimes_R S \oplus Q \otimes_R S] - [P \otimes_R S] - [Q \otimes_R S]$ and so $I \cdot \mathbb{Z}[\mathrm{Proj}_R] = I$ as desired. Thus $K_0(R) = \mathbb{Z}[\mathrm{Proj}_R]/I$ is a ring, being the quotient of a ring by an ideal. It is also commutative because $P \otimes_R Q \cong Q \otimes_R P$.

   (b): Suppose $f : R \to S$ is a ring homomorphism. If P is a projective R module then $P \otimes_R S$ is a projective S module: indeed, if $P \oplus M = F$ then $P \otimes_R S \oplus M \otimes_R S = F \otimes_R S$ which is free. Consider $\sum a_i[P_i] \mapsto \sum a_i[P_i \otimes_R S]$. This is easily seen to be a ring homomorphism $\mathbb{Z}[\mathrm{Proj}_R] \to \mathbb{Z}[\mathrm{Proj}_S]$. Since $- \otimes_R S$ takes direct sums to direct sums it follows that we get a well-defined ring homomorphism $K_0(R) \to K_0(S)$. We conclude easily now that K_0 is a functor.

   (c): Recall from last semester that every finitely generated projective module over a PID is free. Thus $P \cong R^n$ and so $\mathbb{Z}[\mathrm{Proj}_R] = \mathbb{Z}[R^n | n \geq 0]$. Moreover, $[R^n] = [R \oplus \cdots \oplus R] = n[R]$ in $K_0(R)$ and so $K_0(R) = \mathbb{Z} \cdot [R]$ as desired. □

4. Let R be a ring. Consider the following commutative diagram of R-module homomorphisms with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

   Show that there exists an exact sequence

$$0 \to \ker a \to \ker b \to \ker c \to \operatorname{coker} a \to \operatorname{coker} b \to \operatorname{coker} c \to 0$$

   This is known as the snake lemma.

Proof. Consider the columns as complexes $A^\bullet$, $B^\bullet$ and $C^\bullet$. Then the hypothesis is that the sequence $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ is an exact sequence of complexes. The long exact sequence for cohomology of complexes yields $H^{-1}(C^\bullet) \to H^0(A^\bullet) \to H^0(B^\bullet) \to H^0(C^\bullet) \to H^1(A^\bullet) \to H^1(B^\bullet) \to H^1(C^\bullet) \to H^2(A^\bullet)$. Then Example 54 from the lecture notes yields the snake lemma. $\square$

5. (a) Let $\mathcal{C}$ be the category of local Noetherian commutative rings R such that $R/\mathfrak{m}_R \cong \mathbb{Q}$ and morphisms $f : R \to S$ such that $f(\mathfrak{m}_R) = \mathfrak{m}_S$. Let V be an n-dimensional rational vector space and $T \in \mathrm{End}_{\mathbb{Q}}(V)$. By a *deformation* of (V, T) to $R \in \mathrm{Ob}(\mathcal{C})$ we mean a free R-module V_R of rank n and $T_R \in \mathrm{End}_R(V_R)$ such that $(V_R, T_R) \otimes_R (R/\mathfrak{m}_R, 1) \cong (V, T)$. Show that sending $R \in \mathrm{Ob}(\mathcal{C})$ to the set of deformations of (V, T) to R yields a functor $D : \mathcal{C} \to \mathrm{Sets}$.

   (b) Let $\phi : R \to S$ be a homomorphism of commutative rings giving S the structure of an R-algebra. Let M be an S-module. Let $\mathrm{Der}_R(S, M)$ be the set of R-module homomorphisms $d : S \to M$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in S$. Show that $\mathrm{Der}_R(S, -)$ gives a covariant functor from S-modules to R-modules.

Proof. (a): Clearly D sends rings to sets so we need only construct D of morphisms $\mathcal{C}$. Suppose $f : R \to S$ is a morphism in $\mathcal{C}$. For each deformation $(V_R, T_R) \in D(R)$ define $(V_S, T_S) = (V_R, T_R) \otimes_R (S, \mathrm{id}_S)$ where S is an R-module via f. We only need to show that $(V_S, T_S) \in D(S)$ because then $D(R)$ maps to $D(S)$ via $\otimes_R (S, \mathrm{id}_S)$ and thus D is a functor. First, if $V_R \cong R^n$ then $V_S \cong S^n$. Next, we need to check that $(V_S, T_S) \otimes_S (S/\mathfrak{m}_S, \mathrm{id}) \cong (V, T)$. But

$$\begin{aligned}
(V_S, T_S) \otimes_S (S/\mathfrak{m}_S, \mathrm{id}) &\cong (V_S \otimes_S (S/\mathfrak{m}_S), T_S \otimes \mathrm{id}) \\
&= ((V_R \otimes_R S) \otimes (S/\mathfrak{m}_S), (T_R \otimes \mathrm{id}) \otimes \mathrm{id} \\
&\cong (V_R \otimes_R (S \otimes_S S/\mathfrak{m}_S), T_R \otimes (\mathrm{id} \otimes \mathrm{id})) \\
&\cong (V_R \otimes_R S/\mathfrak{m}_S, T_R \otimes \mathrm{id}) \\
&\cong (V_R \otimes R/\mathfrak{m}_R, T_R \otimes \mathrm{id}) \\
&\cong (V, T)
\end{aligned}$$

where we used in row 3 that if N is a bi-(A, B)-module then $(M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P)$ and in row 5 the fact that f induced the unique isomorphism $R/\mathfrak{m}_R \cong \mathbb{Q} \cong S/\mathfrak{m}_S$. (Here we used that $\mathbb{Q}$ as a ring has a unique isomorphism.)

(b): Note that if $d \in \mathrm{Der}_R(S, M)$ then for $r \in R$, rd is also a derivation. Also two derivations can be added as functions and one obtains again a derivation. Thus $\mathrm{Der}_R(S, M)$ is an R-module. Suppose $f : M \to N$ is a morphism of S-modules. If $d \in \mathrm{Der}_R(S, M)$ then $f \circ d : S \to N$. If $x, y \in S$ then $f(d(xy)) = f(xdy + ydx) = f(xdy) + f(ydx) = xf(dy) + yf(dx)$ as f is an S-module morphism. Thus $f \circ d \in \mathrm{Der}_R(S, N)$. It is now trivial to check that $\mathrm{Der}_R(S, -)$ yields a functor.

$\square$