## Graduate Algebra Homework 4

## Due 2015-02-18

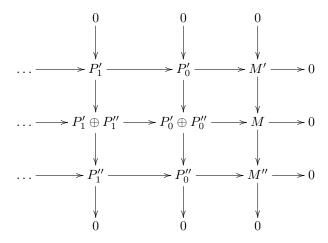
*Remark* 1. It happened in the past that some problems were perceived as much more complicated than what they really were. As an added help, when there is a problem that is straightforward I will put an asterisk next to it.

- 1. Let R be a commutative local ring with maximal ideal  $\mathfrak{m}$ .
  - (a) (Optional) Let M be a finitely generated R-module. Suppose  $m_1, \ldots, m_k \in M$  such that  $m_1, \ldots, m_k \mod \mathfrak{m}$  form a basis of  $M/\mathfrak{m}M$  over the field  $R/\mathfrak{m}$ . Show that  $m_1, \ldots, m_k$  generate M as an R-module. [Hint: Nakayama's lemma.]
  - (b) If M is a finitely generated projective R-module, show that M is free. [Hint: Show that M is a direct summand of a finite rank free R module. Then use (a).]
- 2. Let R be a commutative ring.
  - (a) \* Show that every finitely generated projective *R*-module *N* is locally free, i.e.,  $N_{\mathfrak{p}}$  is free over  $S_{\mathfrak{p}}$  for any prime ideal  $\mathfrak{p}$  of *S*.
  - (b) (Optional) Suppose S is an R-algebra and M is an R-module. Show that  $S \otimes_R \wedge^k M \cong \wedge^k (S \otimes_R M)$  for all  $k \ge 0$ . Conclude that formation of exterior powers commutes with localizations.
  - (c) Show that if M, N are finitely generated projective R-modules then

$$\wedge^k (M \oplus N) \cong \bigoplus_{i+j=k} \wedge^i M \otimes_R \wedge^j N$$

[Hint: Take the natural map from the RHS to the LHS. To check that this is an isomorphism you may use that being an isomorphism is a local property. Then use (b).]

3. Let R be a ring and  $0 \to M' \to M \to M'' \to 0$  an exact sequence of R-modules. Suppose  $\ldots \to P'_1 \to P'_0 \to M' \to 0$  and  $\ldots \to P''_1 \to P''_0 \to M'' \to 0$  are two projective resolutions. Show that there exist R-module maps such that the following diagram is commutative with exact rows and columns:



[Hint: Use the snake lemma to construct the maps inductively.]

- 4. Let R be a commutative ring, S a commutative R-algebra, M an S-module and N an R-module.
  - (a) \* Show that  $\operatorname{Hom}_R(M, N)$  is an S-module with respect to  $(s \cdot f)(m) = f(sm)$ .
  - (b) Consider the map  $\operatorname{Hom}_R(M, N) \to \operatorname{Hom}_S(M, \operatorname{Hom}_R(S, N))$  sending f to  $m \mapsto (s \mapsto f(sm))$ . Show that this is an isomorphism of S-modules.
  - (c) \* If I is an injective R-module show that  $\operatorname{Hom}_R(S, I)$  is an injective S-module.
  - (d) If R is a field show that M is injective as an R-module and conclude that M, as an S-module, injects into an injective S-module.

This exercise is true for non-commutative algebras too, with care taken about left and right modules.