Graduate Algebra
Homework 4

Due 2015-02-18

Remark 1. It happened in the past that some problems were perceived as much more complicated than what they really were. As an added help, when there is a problem that is straightforward I will put an asterisk next to it.

1. Let R be a commutative local ring with maximal ideal $\mathfrak{m}$.
   (a) (Optional) Let M be a finitely generated R-module. Suppose $m_1, \ldots, m_k \in M$ such that $m_1, \ldots, m_k \mod \mathfrak{m}$ form a basis of $M/\mathfrak{m}M$ over the field $R/\mathfrak{m}$. Show that $m_1, \ldots, m_k$ generate M as an R-module. [Hint: Nakayama's lemma.]
   (b) If M is a finitely generated projective R-module, show that M is free. [Hint: Show that M is a direct summand of a finite rank free R module. Then use (a).]

2. Let R be a commutative ring.
   (a) * Show that every finitely generated projective R-module N is locally free, i.e., $N_{\mathfrak{p}}$ is free over $S_{\mathfrak{p}}$ for any prime ideal $\mathfrak{p}$ of S.
   (b) (Optional) Suppose S is an R-algebra and M is an R-module. Show that $S \otimes_R \wedge^k M \cong \wedge^k (S \otimes_R M)$ for all $k \geq 0$. Conclude that formation of exterior powers commutes with localizations.
   (c) Show that if M, N are finitely generated projective R-modules then
   $$\wedge^k (M \oplus N) \cong \bigoplus_{i+j=k} \wedge^i M \otimes_R \wedge^j N$$
   [Hint: Take the natural map from the RHS to the LHS. To check that this is an isomorphism you may use that being an isomorphism is a local property. Then use (b).]

3. Let R be a ring and $0 \to M' \to M \to M'' \to 0$ an exact sequence of R-modules. Suppose $\ldots \to P_1' \to P_0' \to M' \to 0$ and $\ldots \to P_1'' \to P_0'' \to M'' \to 0$ are two projective resolutions. Show that there exist R-module maps such that the following diagram is commutative with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P_1' & \longrightarrow & P_0' & \longrightarrow & M' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P_1' \oplus P_1'' & \longrightarrow & P_0' \oplus P_0'' & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P_1'' & \longrightarrow & P_0'' & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

[Hint: Use the snake lemma to construct the maps inductively.]

4. Let R be a commutative ring, S a commutative R-algebra, M an S-module and N an R-module.

   (a) * Show that $\mathrm{Hom}_R(M, N)$ is an S-module with respect to $(s \cdot f)(m) = f(sm)$.

   (b) Consider the map $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_S(M, \mathrm{Hom}_R(S, N))$ sending f to $m \mapsto (s \mapsto f(sm))$. Show that this is an isomorphism of S-modules.

   (c) * If I is an injective R-module show that $\mathrm{Hom}_R(S, I)$ is an injective S-module.

   (d) If R is a field show that M is injective as an R-module and conclude that M, as an S-module, injects into an injective S-module.

   (e) (Optional) If $R = \mathbb{Z}$ (and every ring is a $\mathbb{Z}$-algebra), show that $\mathbb{Q}/\mathbb{Z}$ is an injective R-module and conclude that M, as an S-module, injects into an injective S-module. [Hint: Show that as a $\mathbb{Z}$-module M injects into $\prod_{f \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}$.]

   This exercise is true for non-commutative algebras too, with care taken about left and right modules.