Graduate Algebra
Homework 4

Due 2015-02-18

1. Let R be a commutative local ring with maximal ideal $\mathfrak{m}$.

   (a) (Optional) Let M be a finitely generated R-module. Suppose $m_1, \ldots, m_k \in M$ such that $m_1, \ldots, m_k \mod \mathfrak{m}$ form a basis of $M/\mathfrak{m}M$ over the field $R/\mathfrak{m}$. Show that $m_1, \ldots, m_k$ generate M as an R-module. [Hint: Nakayama's lemma.]

   (b) If M is also projective, show that M is free. [Hint: Find a suitable free module of which M is a direct summand.]

   Proof. (b): Let $m_1, \ldots, m_k$ be a minimal set of generators of M over R. This gives an exact sequence $R^k \to M \to 0$ and let K be the kernel. Thus $0 \to K \to R^k \to M \to 0$. Since M is projective this sequence splits as $R^k = M \oplus K$. Tensoring with $R/\mathfrak{m}$ we get $(R/\mathfrak{m})^k = M/\mathfrak{m}M \oplus N/\mathfrak{m}N$. By part (a) any basis of $M/\mathfrak{m}M$ as a vector space over the field $R/\mathfrak{m}$ lifts to a set of generators of M. Thus $\dim_{R/\mathfrak{m}} M/\mathfrak{m}M \geq k$ and thus must equal k. This implies that $N/\mathfrak{m}N = 0$. Since $\mathfrak{m}$ is the Jacobson radical it follows from Nakayama's lemma that $N = 0$ and so M is free. □

2. Let R be a commutative ring.

   (a) Show that every finitely generated projective R-module N is locally free, i.e., $N_\mathfrak{p}$ is free over $S_\mathfrak{p}$ for any prime ideal $\mathfrak{p}$ of S.

   (b) (Optional) Suppose S is an R-algebra and M is an R-module. Show that $S \otimes_R \wedge^k M \cong \wedge^k (S \otimes_R M)$ for all $k \geq 0$. Conclude that formation of exterior powers commutes with localization.

   (c) Show that if M, N are finitely generated projective R-modules then

   $$\wedge^k (M \oplus N) \cong \bigoplus_{i+j=k} \wedge^i M \otimes_R \wedge^j N$$

   [Hint: Whether a morphism of modules is an isomorphism is a local property.]

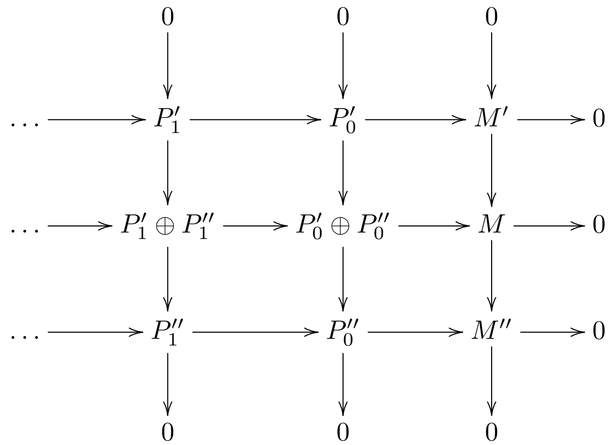
   Proof. (a): For each prime ideal $\mathfrak{p}$ the ring $R_\mathfrak{p}$ is local. Since N is projective there exists M such that $M \oplus N$ is free of finite rank (see previous exercise). But then $M_\mathfrak{p} \oplus N_\mathfrak{p}$ is free of finite rank and so $N_\mathfrak{p}$ is finitely generated projective. From 1 (b) it follows that $N_\mathfrak{p}$ is free.

   (b): Consider $f : S \otimes_R \wedge^k M \to \wedge^k (S \otimes_R M)$ sending $\sum_i s_i \otimes \wedge_j m_{i,j}$ to $\sum_i s_i \wedge_j (1 \otimes m_{i,j})$. Also let $g : \wedge^k (S \otimes_R M) \to S \otimes_R \wedge^k M$ sending $\sum_j \wedge_i s_{i,j} \otimes m_{i,j}$ to $\sum_j (\prod_i s_{i,j}) \otimes \wedge_i m_{i,j}$. Then f and g are mutually inverse isomorphisms. For a prime ideal $\mathfrak{p}$ setting $S = R_\mathfrak{p}$ we deduce that $\wedge^i M_\mathfrak{p} \cong (\wedge^i M)_\mathfrak{p}$.

   (c): The localization functor is simply $- \otimes_R R_\mathfrak{p}$ and so localization, by (b), commutes with exterior powers. It also commutes with tensor products and direct sums.

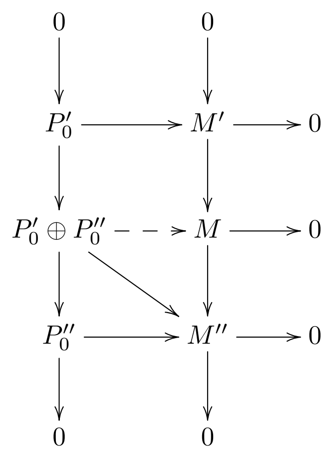
   Consider the homomorphism $f : \bigoplus \wedge^i M \otimes_R \wedge^j N \to \wedge^k (M \oplus N)$ sending $\oplus m_i \otimes n_j$ to $\sum m_i \wedge n_j$. Localizing at $\mathfrak{p}$ we get $f_\mathfrak{p} : \bigoplus \wedge^i M_\mathfrak{p} \otimes_{R_\mathfrak{p}} \wedge^j N_\mathfrak{p} \to \wedge^k (M_\mathfrak{p} \oplus N_\mathfrak{p})$. From class we know that this natural map is an isomorphism. Thus $f_\mathfrak{p}$ is an isomorphism for all $\mathfrak{p}$ which implies that f is an isomorphism. □

3. Let R be a ring and $0 \to M' \to M \to M'' \to 0$ an exact sequence of R-modules. Suppose $\ldots \to P_1' \to P_0' \to M' \to 0$ and $\ldots \to P_1'' \to P_0'' \to M'' \to 0$ are two projective resolutions. Show that there exist R-module maps such that the following diagram is commutative with exact rows and columns:



[Hint: You might find the snake lemma useful.]

Proof. We'll do by induction on n. For $n = 0$ take the vertical maps to be split exact sequence $0 \to P_0' \to P_0' \oplus P_0'' \to P_0'' \to 0$.



where the diagonal arrow is composition and the dashed arrow follows from projectivity of $P_0' \oplus P_0''$. The snake lemma then implies that the dashed arrow is surjective and that we get a commutative diagram with exact columns

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker' & \longrightarrow & P_0' & \longrightarrow & M' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker & \longrightarrow & P_0' \oplus P_0'' & \dashrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & \searrow & \downarrow & & \\
0 & \longrightarrow & \ker'' & \longrightarrow & P_0'' & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Remark that $\ldots \to P_1' \to \ker' \to 0$ and $\ldots P_1'' \to \ker'' \to 0$ are projective resolutions. The above argument implies the existence of a commutative diagram with exact columns

$$
\begin{array}{ccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
P_1' & \longrightarrow & \ker' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
P_1' \oplus P_1'' & \dashrightarrow & \ker & \longrightarrow & 0 \\
\downarrow & \searrow & \downarrow & & \\
P_1'' & \longrightarrow & \ker'' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}
$$

and again the dashed arrow is surjective. This implies that the resulting diagram

$$
\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
P_1' & \longrightarrow & P_0' & \longrightarrow & M' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
P_1' \oplus P_1'' & \longrightarrow & P_0' \oplus P_0'' & \longrightarrow & M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
P_1'' & \longrightarrow & P_0'' & \longrightarrow & M'' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}
$$

has exact rows and columns. Continuing this procedure yields the desired projective resolution of M. $\square$

4. Let R be a commutative ring, S a commutative R-algebra, M an S-module and N an R-module.

   (a) Show that $\mathrm{Hom}_R(M, N)$ is an S-module with respect to $(s \cdot f)(m) = f(sm)$.

   (b) Consider the map $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_S(M, \mathrm{Hom}_R(S, N))$ sending f to $m \mapsto (s \mapsto f(sm))$. Show that this is an isomorphism of S-modules.

   (c) If I is an injective R-module show that $\mathrm{Hom}_R(S, I)$ is an injective S-module.

   (d) If R is a field show that M is injective as an R-module and conclude that M, as an S-module, injects into an S-module.

   (e) (Optional) If $R = \mathbb{Z}$ (and every ring is a $\mathbb{Z}$-algebra), show that $\mathbb{Q}/\mathbb{Z}$ is an injective R-module and conclude that M, as an S-module, injects into an S-module. [Hint: Show that as a $\mathbb{Z}$-module M injects into $\prod_{f \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}$.]

   This exercise is true for non-commutative algebras too, with care taken about left and right modules.

   Proof. (a): $\mathrm{Hom}_R(M, N)$ is an R-module so we only need to check that multiplication by S satisfies the module axioms. The formula implies that $s \cdot (r \cdot f) = (sr) \cdot f$ and R-linearity of f implies distributivity with respect to addition.

   (b): Suppose $g : M \to \mathrm{Hom}_R(S, N)$ is an S-morphism. Define $f : M \to N$ by $f(m) = g(m)(1)$. This is clearly R-linear and the two maps between $\mathrm{Hom}_R(M, N)$ amd $\mathrm{Hom}_S(M, \mathrm{Hom}_R(S, N))$ are inverses to each other, using the formula from (a).

   (c): If I is R-injective then $\mathrm{Hom}_R(-, I)$ is exact and so $\mathrm{Hom}_S(-, \mathrm{Hom}_R(S, I))$ is exact which implies that $\mathrm{Hom}_R(S, I)$ is S-injective.

   (d): Every subvector space A of B is a direct summand so $B = A \oplus A'$. This implies that every homomorphism $A \to M$ lifts to $B \to M$ simply by projecting to A. Thus M is injective. Alternatively R, being a field, is a PID and M is divisible. Consider the identity map $M \to M$. This yields the map $M \to \mathrm{Hom}_R(S, M)$ sending $m \mapsto (s \mapsto sm)$. This is clearly injective (evaluate at $s = 1$) so M injects into the injective S-module $\mathrm{Hom}_R(S, M)$. □