# Graduate Algebra
# Homework 5

Due 2015-02-25

1. Consider the ideal $I = (x, y) \subset R = \mathbb{C}[x, y]$ and $\mathbb{C}$ as the $R$-module $R/I$.
   (a) Show that $\operatorname{Tor}_1^R(I, \mathbb{C}) \cong \operatorname{Tor}_2^R(\mathbb{C}, \mathbb{C})$.
   (b) Find a projective resolution of $\mathbb{C}$. [Hint: Use the algorithm from class.]
   (c) Show that $I$ is not flat over $R$. [Hint: Use (a) and (b).]
   (d) (Optional) Find the kernel of the multiplication map $I \otimes_R I \to I$ as a submodule of $I \otimes_R I$.

2. Suppose $R$ is a commutative ring and $r \in R$. When $r$ is not a zero divisor we saw in class that $\operatorname{Tor}_1^R(R/(r), M) \cong M[r]$ and $\operatorname{Tor}_n^R(R/(r), M) = 0$ for $n \geq 2$. Show that if $r$ is a zero divisor then

$$\operatorname{Tor}_n^R(R/(r), M) \cong \operatorname{Tor}_{n-2}^R(R[r], M)$$

   for $n \geq 3$, where $R[r] = \{s \in R | rs = 0\}$. [Hint: Look at the exact sequence $0 \to R[r] \to R \to R \to R/(r) \to 0$.]

3. Let $R$ be a commutative ring. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of $R$-modules. If $M'$ and $M''$ are flat/injective/projective show that $M$ is flat/injective/projective. [Hint: Use the derived functor criterion for flat/injective/projective.]

4. Let $R$ be a commutative ring and $S$ a commutative $R$-algebra. Recall that you showed that $\operatorname{Der}_R(S, -)$ is a covariant functor from $S$-modules to sets.
   (a) * For $r \in R$ and $d \in \operatorname{Der}_R(S, M)$ show that $d(r) = 0$.
   (b) Consider $S \otimes_R S$ as a ring with respect to coordinate-wise multiplication (i.e., $(a \otimes b) \cdot (a' \otimes b') = (aa') \otimes (bb')$) and as an $S$-module with respect to $s \cdot (a \otimes b) = (sa) \otimes b$. Let $I$ be the kernel of the multiplication $S$-module homomorphism $S \otimes_R S \to S$ and let $\Omega_{S/R} = I/I^2$ as an $S$-module (here $I^2$ is with respect to the ring multiplication on $S \otimes_R S$). Define $D : S \to \Omega_{S/R}$ sending $s \in S$ to $s \otimes 1 - 1 \otimes s$. Show that $D \in \operatorname{Der}_R(S, \Omega_{S/R})$.
   (c) * Let $M$ be an $S$-module. Show that $S * M$ defined as the abelian group $S \oplus M$ together with multiplication $(s, m) \cdot (s', m') = (ss', sm' + s'm)$ is an $S$-algebra which contains $M$ via $m \mapsto (0, m)$ as a sub-$S$-module.
   (d) Let $M$ be as above and $d \in \operatorname{Der}_R(S, M)$. Show that there exists an $S$-algebra homomorphism $\phi : S \otimes_R S \to S * M$ such that $\phi(x \otimes y) = (xy, xd(y))$; show that this homomorphism factors through an $S$-module homomorphism $I/I^2 \to M$.
   (e) Deduce that the functor $\operatorname{Der}_R(S, -)$ is represented by the $S$-module $\Omega_{S/R}$.