Graduate Algebra
Homework 5

Due 2015-02-25

1. Consider the ideal $I = (x, y) \subset R = \mathbb{C}[x, y]$ and $\mathbb{C}$ as the R-module R/I.
   (a) Show that $\mathrm{Tor}_1^R(I, \mathbb{C}) \cong \mathrm{Tor}_2^R(\mathbb{C}, \mathbb{C})$.
   (b) Find a projective resolution of $\mathbb{C}$. [Hint: Use the algorithm from class.]
   (c) Show that I is not flat over R. [Hint: Use (a) and (b).]
   (d) (Optional) Find the kernel of the multiplication map $I \otimes_R I \to I$ as a submodule of $I \otimes_R I$.

   Proof. (a): Consider the exact sequence $0 \to I \to R \to R/I \to 0$ with R in the middle projective. Then dimension shifting gives the desired isomorphism.

   (b): Follow, roughly, the algorithm from class. Have $0 \to I \to R \to R/I \to 0$ but I is not projective. However, there is a surjection $R^2 \to I \to 0$ sending $(a, b) \mapsto ax + by$. The kernel consists of (a, b) such that $ax = -by$ in which case $a = cy$ and $b = -cx$ since $\mathbb{C}[x, y]$ is a UFD. Thus the kernel is isomorphic to R and so $0 \to R \xrightarrow{c \mapsto (cy, -cx)} R^2 \xrightarrow{(a,b) \mapsto ax+by} R \to R/I \to 0$ is an exact sequence with free and therefore projective modules.

   (c): Note that

$$\begin{aligned} \mathrm{Tor}_1(I.\mathbb{C}) &\cong \mathrm{Tor}_2(\mathbb{C}, \mathbb{C}) \\ &\cong H^2(0 \to R \otimes_R \mathbb{C} \to R^2 \otimes_R \mathbb{C} \to R \otimes_R \mathbb{C}) \\ &= \ker(R \otimes_R \mathbb{C} \to R^2 \otimes_R \mathbb{C}) \\ &= R \otimes_R \mathbb{C} \\ &= \mathbb{C} \end{aligned}$$

   since the map $c \otimes z \mapsto (cy, -cx) \otimes z = c \otimes yz - c \otimes xz = 0$ as x, y are 0 in $\mathbb{C} = R/I$. Thus I is not flat.

   (d): We know that $\mathrm{Tor}_1(I, \mathbb{C}) = \mathrm{Tor}_1(I, R/I) = \ker(I \otimes_R I \to I)$ and this is isomorphic to $\mathbb{C}$. Note that $x \otimes y - y \otimes x$ is in the kernel and thus the kernel is $\mathbb{C}(x \otimes y - y \otimes x)$. □

2. Suppose R is a commutative ring and $r \in R$. When r is not a zero divisor we saw in class that $\mathrm{Tor}_1^R(R/(r), M) \cong M[r]$ and $\mathrm{Tor}_n^R(R/(r), M) = 0$ for $n \geq 2$. Show that if r is a zero divisor then

$$\mathrm{Tor}_n^R(R/(r), M) \cong \mathrm{Tor}_{n-2}^R(R[r], M)$$

   for $n \geq 3$, where $R[r] = \{s \in R | rs = 0\}$. [Hint: Look at the exact sequence $0 \to R[r] \to R \to R \to R/(r) \to 0$.]

   Proof. Consider $R \xrightarrow{\times r} R$ with kernel $R[r]$ and cokernel $R/(r)$. Get two exact sequences $0 \to R[r] \to R \to R/R[r] \to 0$ and $0 \to (r) \to R \to R/(r) \to 0$ with $R/R[r] = (r)$ by the first isomorphism theorem. The middle modules are free in both cases so dimension shifting (applied twice) yields

$$\mathrm{Tor}_n(R/(r), M) \cong \mathrm{Tor}_{n-1}((r), M) \cong \mathrm{Tor}_{n-1}(R/R[r], M) \cong \mathrm{Tor}_{n-2}(R[r], M)$$

   □

3. Let R be a commutative ring. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules. If M' and M'' are flat/injective/projective show that M is flat/injective/projective. [Hint: Use the derived functor criterion for flat/injective/projective.]

Proof. If N is any module then the long exact sequence yields

$$\mathrm{Tor}_1(M', N) \to \mathrm{Tor}_1(M, N) \to \mathrm{Tor}_1(M'', N)$$

and if M' and M'' are flat then the outside Tors vanish and so $\mathrm{Tor}_1(M, N) = 0$ for all N. Thus M is flat.

Similarly get

$$\mathrm{Ext}^1_R(N, M') \to \mathrm{Ext}^1_r(N, M) \to \mathrm{Ext}^1_R(N, M'')$$

and

$$\mathrm{Ext}^1_R(M'', N) \to \mathrm{Ext}^1_r(M, N) \to \mathrm{Ext}^1_R(M', N)$$

If M' and M'' are projective/injective then the outside Exts in the second/first sequence vanish and thus the middle Ext vanishes in the second/first sequence. Since this is true for all N we deduce that M is also projective/injective. $\square$

4. Let R be a commutatve ring and S a commutative R-algebra. Recall that you showed that $\mathrm{Der}_R(S, -)$ is a covariant functor from S-modules to sets.

   (a) * For $r \in R$ and $d \in \mathrm{Der}_R(S, M)$ show that $d(r) = 0$.

   (b) Consider $S \otimes_R S$ as a ring with respect to coordinate-wise multiplication (i.e., $(a \otimes b) \cdot (a' \otimes b') = (aa') \otimes (bb')$) and as an S-module with respect to $s \cdot (a \otimes b) = (sa) \otimes b$. Let I be the kernel of the multiplication S-module homomorphism $S \otimes_R S \to S$ and let $\Omega_{S/R} = I/I^2$ as an S-module. Define $D : S \to \Omega_{S/R}$ sending $s \in S$ to $s \otimes 1 - 1 \otimes s$. Show that $D \in \mathrm{Der}_R(S, \Omega_{S/R})$.

   (c) * Let M be an S-module. Show that $S * M$ defined as the abelian group $S \oplus M$ together with multiplication $(s, m) \cdot (s', m') = (ss', sm' + s'm)$ is an S-algebra which contains M via $m \mapsto (0, m)$ as a sub-S-module.

   (d) Let M be as above and $d \in \mathrm{Der}_R(S, M)$. Show that there exists an S-algebra homomorphism $\phi : S \otimes_R S \to S * M$ such that $\phi(x \otimes y) = (xy, xd(y))$; show that this homomorphism factors through an S-module homomorphism $I/I^2 \to M$.

   (e) * Deduce that the functor $\mathrm{Der}_R(S, -)$ is represented by the S-module $\Omega_{S/R}$.

Proof. (a): The map d is R-linear and so $d(r) = rd(1)$ but $d(r) = d(r \cdot 1) = d(r) + rd(1)$ and so $d(r) = 0$ for all $r \in R$.

(b): First note that if $s \in S$ then indeed $D(s) \in I$ since $s \cdot 1 - 1 \cdot s = 0$.

For $x, y \in S$ we compute

$$\begin{aligned} xD(y) + yD(x) &= x(y \otimes 1 - 1 \otimes y) + y(x \otimes 1 - 1 \otimes x) \\ &= (xy) \otimes 1 - x \otimes y + (yx) \otimes 1 - y \otimes x \\ &= D(xy) + (yx) \otimes 1 + 1 \otimes (xy) - x \otimes y - y \otimes x \\ &= D(xy) + (y \otimes 1 - 1 \otimes y)(x \otimes 1 - 1 \otimes x) \\ &= D(xy) + D(x)D(y) \\ &\equiv D(xy) \pmod{I^2} \end{aligned}$$

as $D(x), D(y) \in I$. Since $\Omega_{S/R} = I/I^2$ we get $xD(y) + yD(x) = D(xy)$.

(c): First, $(1,0)\cdot(s,m)=(s,m)$. Next, $(a+b,m+n)\cdot(s,p)=((a+b)s,(a+b)p+s(m+n))=(as,ap+sm)+(bs,bp+sn)$ which yields distributivity. For associativity we check that

$$\begin{aligned}(a,m)\cdot((b,n)\cdot(c,p)) &= (a,m)\cdot(bc,bp+cn)\\ &= (abc,a(bp+cn)+bcm)\\ &= (ab,an+bm)\cdot(c,p)\\ &= ((a,m)\cdot(b,n))\cdot(c,p)\end{aligned}$$

The ring $S*M$ contains S via $s\mapsto(s,0)$ which gives it the structure of an S-algebra.

Finally, $(s,0)\cdot(0,m)=(0,sm)$ and so M inside $S*M$ is a sub-S-module isomorphic to the S-module M.

(d): Define $\phi(\sum x_i\otimes y_i)=\sum x_i d(y_i)$. We only need to check that this yields and S-algebra homomorphism, i.e., that

$$\phi((x\otimes y)\cdot(x'\otimes y'))=\phi(x\otimes y)\phi(x'\otimes y')$$

But

$$\begin{aligned}\phi((x\otimes y)\cdot(x'\otimes y')) &= \phi((xx')\otimes(yy'))\\ &= (xx'yy',xx'd(yy'))\\ &= (xx'yy',xx'yd(y')+xx'y'd(y))\\ &= (xy,xd(y))\cdot(x'y',x'd(y'))\\ &= \phi(x\otimes y)\phi(x'\otimes y')\end{aligned}$$

Note that the image of I under ϕ lands in the image of M in $S*M$ since $\phi(\sum x_i\otimes y_i)=(\sum x_iy_i,\sum x_id(y_i))$ and $\sum x_iy_i=0$ for $\sum x_i\otimes y_i\in I$ by definition. Thus we get $\phi: I\to M$. Finally, it suffices to check that $\phi(I^2)=0$ to conclude that ϕ factors through $I/I^2\to M$. But ϕ is a ring-homomorphism so $\phi(I^2)=\phi(I)^2\subset M\cdot M=0$ since $(0,m)\cdot(0,n)=(0,0)$.

(e): We show that $\mathrm{Der}_R(S,-)$ is represented by $\Omega_{S/R}$ and $D\in\mathrm{Der}_R(S,\Omega_{S/R})$. We need to check that for each S-module M we have a bijection

$$\begin{aligned}\mathrm{Hom}_S(\mathcal{O}_{S/R},M)&\to\mathrm{Der}_R(S,M)\\ \phi&\mapsto\phi\circ D\end{aligned}$$

The map LHS to RHS makes sense since $\mathrm{Der}_R(S,-)$ is a functor. For surjectivity take $d\in\mathrm{Der}_R(S,M)$. Then part (d) yields $\phi_d:\Omega_{S/R}\to M$ in the LHS for which

$$\phi\circ D(s)=\phi(s\otimes 1-1\otimes s)=(s,sd(1))-(s,d(s))=(s,0)-(s,d(s))=(0,-d(s))$$

so $\phi\circ D=-d$ which gives surjectivity.

Finally, for injectivity, suppose that $\phi\circ D=\phi'\circ D$. Then $(\phi-\phi')\circ D=0$ so it's enough to show that if $\phi\circ D=0$ then $\phi=0$. Note that

$$\sum x_i\otimes y_i=(\sum x_iy_i)\otimes 1-\sum x_iD(y_i)$$

and so if $\sum x_i\otimes y_i\in I$ then $\sum x_i\otimes y_i=-\sum x_iD(y_i)$. But then

$$\phi(\sum x_i\otimes y_i)=-\sum\phi(x_iD(y_i))=-\sum x_i\phi(D(y_i))=0$$

as desired. □