# Graduate Algebra
# Homework 6

Due 2015-03-04

1. Let $n \geq 2$. For a ring $R$ define $\mathrm{GL}(n, R)$ as the set of $n \times n$ matrices $M$ such that $M^{-1}$ is also in $M_{n\times n}(R)$.
   (a) Show that $\mathrm{GL}(n, R) = \{g \in M_{n\times n}(R) | \det(M) \in R^\times\}$.
   (b) Show that $\mathrm{GL}(n, -)$ yields a covariant functor from the category of rings (with morphisms taking 1 to 1) to the category of sets.
   (c) Show that $\mathrm{GL}(n, -)$ is representable.
2. (a) Suppose $L/K$ is a field extension such that $L$ has $p^n$ elements and $K$ has $p^m$ elements. Show that $m \mid n$.
   (b) Suppose $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ where $\alpha_i^2 \in \mathbb{Q}$ for $1 \leq i \leq n$. Show that $\sqrt[3]{2} \notin K$. [Hint: The degree is multiplicative in towers of extensions.]
3. In each of the following examples you are given a polynomial $P(X) \in K[X]$ over some field $K$. In each case find the splitting field of $P$ over $K$ as well as the degree over $K$ of the splitting field. The letter $p$ denotes a prime number.
   (a) $X^p - 2 \in \mathbb{Q}[X]$.
   (b) $X^{p-1} - t \in \mathbb{F}_p(t)[X]$ for $p > 2$.
   (c) $X^4 + X^2 + 1 \in \mathbb{Q}[X]$.
   (d) $X^n - t - 1 \in \mathbb{C}(\!(t)\!)[X]$. Here $\mathbb{C}(\!(t)\!)$ is the fraction field of $\mathbb{C}[\![t]\!]$ consisting of Laurent series.
4. Let $K$ be a field and $K(x)$ be the field of rational functions with coefficients in $K$. Let $P(x), Q(x) \in K[x]$ be two coprime polynomials and $t = P/Q \in K(x)$.
   (a) Show that $P(X) - tQ(X) \in K(t)[X]$ is irreducible and has $X = x$ as a root. [Hint: Use Gauss' lemma and the fact that $K[X][t] = K[t][X]$.]
   (b) Conclude that $[K(x) : K(t)] = \max(\deg(P), \deg(Q))$.
5. Suppose $L/K$ is a finite extension of fields and $K \subset M_1, M_2 \subset L$ are two subextensions. Show that $M_1 \otimes_K M_2$ is a field if and only if $[M_1M_2 : K] = [M_1 : K][M_2 : K]$. [Hint: Look at the multiplication map $M_1 \otimes_K M_2 \to M_1M_2$.]