Graduate Algebra
Homework 7

Due 2015-04-01

1. Let α and β be elements of a finite extension L/K.
   (a) If $[K(\alpha) : K]$ is odd show that $K(\alpha) = K(\alpha^2)$.
   (b) If the degree of the minimal polynomials $P_\alpha(X)$ (of α over K) and $P_\beta(X)$ (of β over K) are coprime show that $P_\alpha(X)$ is irreducible over $K(\beta)$.
   (c) If K has characteristic p which does not divide $[L : K]$ show that α is separable over K.
2. Let L/K be a finite extension and K_1, K_2 be two subextensions of K such that $K_1 \cap K_2 = K$ and $[K_2 : K] = 2$. Show that $[K_1K_2 : K] = [K_1 : K][K_2 : K]$.
3. Suppose K is not perfect. Show that there exist inseparable irreducible polynomials in $K[X]$.
4. Let $\alpha = \sqrt[4]{5}$.
   (a) Is $\mathbb{Q}(i\alpha^2)$ normal over $\mathbb{Q}$?
   (b) Is $\mathbb{Q}(\alpha + i\alpha)$ normal over $\mathbb{Q}(i\alpha^2)$?
   (c) Is $\mathbb{Q}(\alpha + i\alpha)$ normal over $\mathbb{Q}$?
5. Let p be a prime and $\alpha \in \mathbb{F}_p^\times$.
   (a) Let $Q(X) = X^p - X - a$. Show that $Q(X + 1) = Q(X)$.
   (b) Show that the splitting field K of Q over $\mathbb{F}_p$ is a normal separable extension of degree p. [Hint: Use (a).]
   (c) Determine the set $\mathrm{Aut}(K/\mathbb{F}_p)$. [Hint: Use (a).]

   K is an Artin-Schreier extension.
6. Suppose $\sigma \in \mathrm{Aut}(\mathbb{R}/\mathbb{Q})$.
   (a) Show that if $x > 0$ then $\sigma(x) > 0$ and conclude that σ is an increasing function.
   (b) Show that if $|x - y| < \frac{1}{n}$ then $|\sigma(x) - \sigma(y)| < \frac{1}{n}$ and conclude that σ is continuous.
   (c) Show that $\mathrm{Aut}(\mathbb{R}/\mathbb{Q}) = \{\mathrm{id}\}$.
7. Let K be any field and x a variable. Recall that $\mathrm{PGL}(2, K)$ is the quotient $\mathrm{GL}(2, K)/K^\times I_2$ of invertible 2×2 matrices by the normal subgroup of scalar matrices. Show that as sets

$$\mathrm{Aut}(K(x)/K) \cong \mathrm{PGL}(2, K)$$

   via $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2, K)$ mapping to the automorphism $\sigma_\gamma(f(x)) = f\left(\frac{ax+b}{cx+d}\right)$. [Hint: If $\sigma \in \mathrm{Aut}(K(x)/K)$ then $K(x) = K(\sigma(x))$. What does $\sigma(x)$ look like?]