Graduate Algebra
Homework 8

Due 2015-04-08

1. Suppose n is an odd integer. Show that $\Phi_{2n}(X) = \Phi_n(-X)$.

2. Let $K = \mathbb{Q}(i, \sqrt[8]{2})$ be the splitting field of $X^8 - 2$ over $\mathbb{Q}$. Show that $\mathrm{Gal}(K, \mathbb{Q}(i))$ is the cyclic group $\mathbb{Z}/8\mathbb{Z}$, $\mathrm{Gal}(K/\mathbb{Q}(\sqrt{2})) \cong D_8$ and $\mathrm{Gal}(K/\mathbb{Q}(i\sqrt{2})) \cong Q_8$. [Hint: You might find your job easier if you recall presentations for these groups.]

3. Let $p > 2$ be a prime and g a generator of $\mathbb{F}_p^\times$. Show that the subextensions $\mathbb{Q}(\zeta_p)/K/\mathbb{Q}$ are all of the form
$$K_r = \mathbb{Q}(\sum_{i=1}^{(p-1)/r} \zeta_p^{g^{ri}})$$
where r ranges over the divisors of $p-1$. [Hint: Compute the Galois group of $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/K_r)$. A straightforward problem.]