Graduate Algebra
Homework 10

Due 2015-04-22

1. (a) Let p be a prime and $n \geq 1$. Show that there exists a subextension $\mathbb{Q}(\zeta_{p^{n+2}})/K/\mathbb{Q}$ with $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/p^n\mathbb{Z}$.
   (b) Let G be any finite abelian group. Show there exists a Galois extension $K/\mathbb{Q}$ with $\mathrm{Gal}(K/\mathbb{Q}) \cong G$.

2. (a) Show that the discriminant of the polynomial $X^n + pX + q$ is

$$(-1)^{\binom{n}{2}} n^n q^{n-1} + (-1)^{\binom{n-1}{2}} (n-1)^{n-1} p^n$$

   (b) If $p > 2$ is a prime show that $\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p}) \subset \mathbb{Q}(\zeta_p)$. [Hint: Compute the discriminant of $X^p - 1$.]

3. (a) Let $P(X) \in \mathbb{Q}[X]$ be irreducible with prime degree q and exactly two nonreal roots. Show that P has Galois group S_q. [Hint: S_q is generated by a transposition and a q-cycle.]
   (b) Compute the Galois group of $X^7 + X + 13 \in \mathbb{Q}[X]$. [Hint: You are welcome to use Wolfram Alpha for factorizations. Also, the answer is nice.]

4. Let $L/K/\mathbb{Q}$ be finite extensions and denote by R and S the integral closure of $\mathbb{Z}$ in K and L respectively.
   (a) Show that $\mathrm{Tr}_{L/K} : L \to K$ restricts to $\mathrm{Tr}_{L/K} : S \to R$.
   (b) Show that $\mathcal{ID} = \{x \in L | \mathrm{Tr}_{L/K}(xS) \subset R\}$ is an S-submodle of L and that $\mathcal{D} = \{x \in S | x\mathcal{ID} \subset S\}$ is an ideal of S.
   (c) Suppose $K = \mathbb{Q}$ and so $R = \mathbb{Z}$. Also suppose that $L = \mathbb{Q}(\alpha)$ and $S = \mathbb{Z}[\alpha]$ and let $m_\alpha(X) \in \mathbb{Z}[X]$ be its minimal polynomial over $\mathbb{Z}$, of degree d.
      i. Show that $\dfrac{1}{m_\alpha(X)} \in X^{-d}(1 + X^{-1}\mathbb{Z}[\![X^{-1}]\!])$.
      ii. Show that $\dfrac{1}{m_\alpha(X)} = \displaystyle\sum_{i=1}^{d} \frac{1}{m'_\alpha(\alpha_i)(X - \alpha_i)}$ where $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ are the roots of $m_\alpha(X)$.
      Conclude that $\dfrac{1}{m_\alpha(X)} = \displaystyle\sum_{n \geq 1} X^{-n} \mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}\left(\frac{\alpha^{n-1}}{m'_\alpha(\alpha)}\right)$.
      iii. Show that

$$\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}\left(\frac{\alpha^n}{m'_\alpha(\alpha)}\right) = \begin{cases} 0 & 0 \leq n < d-1 \\ 1 & n = d-1 \\ \in \mathbb{Z} & n \geq d \end{cases}$$

      [Hint: One line proof.]
      iv. Deduce that $m'_\alpha(\alpha) \in \mathcal{D}$. (One can actually show that $\mathcal{D}$ is generated by $m'_\alpha(\alpha)$.) [Hint: Use (iii).]

It turns out that the ideal $\mathcal{D}$ is the annihilator of the module $\Omega^1_{S/R}$ that you studied in a previous homework. It measures ramification.