Graduate Algebra
Homework 11

Due 2015-04-29

1. Let $\mathbb{C}(x^{1/\infty}, y^{1/\infty}) = \cup_{m,n\geq 1}\mathbb{C}(x^{1/m}, y^{1/n})$.
   (a) Show that $\mathbb{C}(x^{1/\infty}, y^{1/\infty})$ is Galois over $\mathbb{C}(x, y)$.
   (b) Compute $\mathrm{Gal}(\mathbb{C}(x^{1/\infty}, y^{1/\infty})/\mathbb{C}(x, y))$.

2. Let L/K be a Galois extension and let $\{M_k|k \in I\}$ be a collection of subextensions $L/M_k/K$ such that M_k/K is finite Galois and $L = \bigcup M_k$. Show that $\mathrm{Gal}(L/K) \cong \varprojlim \mathrm{Gal}(M_k/K)$.

3. Suppose $L_1, L_2/K$ are two (possibly infinite) Galois extensions. Show that L_1L_2/K and $L_1 \cap L_2/K$ are Galois and

$$\mathrm{Gal}(L_1L_2/K) \cong \{(\sigma, \tau) \in \mathrm{Gal}(L_1/K) \times \mathrm{Gal}(L_2/K)|\sigma|_{L_1\cap L_2} = \tau|_{L_1\cap L_2}\}$$

   [Hint: Use the previous problem.]

4. Show that $H^n(\mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q), \mathbb{F}_{q^d}^\times) = 0$ if $n \geq 1$.

5. Let $H \subset G$ be finite groups and N an H-module.
   (a) Let $\mathrm{Ind}_H^G N = \{f : G \to N|f(hg) = h(f(g)), \forall g \in G, h \in H\}$. For $g \in G$ and $f \in \mathrm{Ind}_H^G N$ define $g(f) : G \to N$ by $g(f)(x) = f(xg)$. Show that this yields an action on $\mathrm{Ind}_H^G N$ which turns $\mathrm{Ind}_H^G N$ into a G-module.
   (b) Thinking of N as a $\mathbb{Z}[H]$-module and $\mathrm{Ind}_H^G N$ as a $\mathbb{Z}[G]$-module show that $\mathrm{Ind}_H^G N \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N$ as $\mathbb{Z}[G]$-modules. Here $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N$ is a $\mathbb{Z}[G]$-module via the scalar multiplication $[g]([h] \otimes n) = [gh] \otimes n$. [Hint: Show that the map $f \mapsto \sum_{g\in H\backslash G}[g^{-1}] \otimes f(g)$ is well-defined and yields the isomorphism.]
   (c) If M is a G-module show that $\mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathrm{Ind}_H^G N) \cong \mathrm{Hom}_{\mathbb{Z}[H]}(M, N)$. [Hint: Take $f : M \to \mathrm{Ind}_H^G N$ to $m \mapsto f(m)(1)$ and $\phi : M \to N$ to $m \mapsto (g \mapsto \phi(g(m)))$.]