Math 30810 Honors Algebra 3
Homework 1

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Due Wednesday, August 31

Do any 4 of the following 5 questions. They're all about matrices. Artin refers to the textbook.

1. Artin 1.9 on page 32.

   Proof. (a): $(A+B)(A-B) = A^2 - AB + BA - B^2$ is $A^2 - B^2$ if and only if $AB = BA$, i.e., if the matrices commute.

   (b): $(A+B)^3 = (A+B)^2(A+B) = (A^2+AB+BA+B^2)(A+B) = A^3 + ABA + BA^2 + B^2A + A^2B + AB^2 + BAB + B^3$. □

2. Artin 1.13 on page 32. [Hint: Do you know a good formula for $\frac{1}{1+x}$ where x is a real number with $|x| < 1$?]

   Proof. The formula in question is $1/(1+x) = 1 - x + x^2 - x^3 + \cdots$. The idea is that a nilpotent matrix A satisfies $A^k = 0$ for some k and therefore $A^n = A^k A^{n-k} 0$ for all $n \geq k$ as well. Formally we'd have $1 - A + A^2 - \cdots = 1 - A + A^2 - \cdots + (-1)^{k-1}A^{k-1}$ which is a finite sum. But this formal expression is not enough. However, we can check that in fact $1 - A + A^2 - \cdots + (-1)^{k-1}A^{k-1} = (1+A)^{-1}$ by computing

   $$(1-A+A^2-\cdots+(-1)^{k-1}A^{k-1})(1+A) = 1-A+A^2-\cdots+(-1)^{k-1}A^{k-1}+A-A^2+A^3-\cdots+(-1)^{k-1}A^k$$
   $$= 1 + (-1)^{k-1}A^k = 1$$

   □

3. Artin 6.2 on page 34.

   Proof. First, if A is invertible with A^{-1} having integer entries then $\det(A^{-1}) \in \mathbb{Z}$. But then $1 = \det(I_n) = \det(A)\det(A^{-1})$ which means that $\det(A)$ and $\det(A^{-1})$ are integers with product 1 so they're either both 1 or both -1.

   Now suppose that $\det(A) = \pm 1$. Remember from linear algebra the cofactor matrix A^* whose entries are given by determinants of minors of A where you remove a row and a column. The adjoint matrix satisfies $AA^* = \det(A)I_n$. In the textbook the cofactor matrix is defined in (1.6.7) and the formula I mentioned is Theorem 1.6.9. This is one way you can compute inverses of matrices algorithmically.

   Immediately from the formula, $A^{-1} = (\det(A))^{-1}A^*$. If $\det(A) = \pm 1$ then $(\det(A))^{-1} = \pm 1$. Also, the entries of the cofactor matrix are determinants of minors with integer entries so A^* has integer entries and therefore so does A^{-1}. □

4. Artin M.4 on page 35.

Proof. Remember that if $A = (a_{ij})$ and $B = (b_{ij})$ then $\text{Tr}(AB) = \sum_{i,j} a_{ij}b_{ji}$ and thus $\text{Tr}(AB) = \text{Tr}(BA)$ from the symmetry of the formula. This is, in effect, a solution for problem M.3 in the textbook.

Then $\text{Tr}(AB - BA) = \text{Tr}(AB) - \text{Tr}(BA) = 0$ whereas $\text{Tr}(I_n) = n$. The equation therefore has no solutions. □

5. Suppose A and B are two $n \times n$ matrices with complex entries and X is an indeterminate variable.
   (a) If B is invertible show that $\det(X - AB) = \det(X - BA)$ as degree n monic polynomials in X.
   (b) Show that $\det(X - AB) = \det(X - BA)$ even if B is not invertible. [Hint: Apply part (a) to $B + aI_n$ for a suitable complex number a.]

Proof. (a): Since X is an indeterminate variable it commutes with matrices. When B is invertible $XB^{-1} = B^{-1}X$ and so we compute

$$\begin{aligned}\det(X - AB) &= \det((XB^{-1} - A)B) \\ &= \det(XB^{-1} - A)\det(B) \\ &= \det(B)\det(B^{-1}X - A) \\ &= \det(X - BA)\end{aligned}$$

as desired. □

(b): When B is not invertible, we will perturb B a little and then use some calculus. Let's replace B by $B + yI_n$ for a complex number y. We'd like to apply part (a) so we'd like $B + yI_n$ to be invertible. In other words we'd like $\det(B + yI_n)$ to be invertible. But $\det(B + yI_n)$ is a monic polynomial in y of degree n so surely whenever you pick y not a root of this polynomial, the matrix $B + yI_n$ is invertible.

From part (a) we now know that

$$\det(X - A(B + yI_n)) = \det(X - (B + yI_n)A)$$

for any one of these y. But simply by looking at the formula for determinants we see that both sides of this equality are polynomials in y and these polynomials take equal values for infinitely many y. But then the two polynomials must be equal (a polynomial can have only finitely many roots after all). If the polynomials themselves are equal, it means that we can plug in ANY value of y in the equality and letting $y = 0$ gives part (b).