# Math 30810 Honors Algebra 3
# Homework 2

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Due Thursday, September 8

**Do any 8 of the following 10 questions. Artin a.b.c means chapter a, section b, exercise c.**

1. Artin 2.1.1 on page 69.
2. Artin 2.2.2 on page 69.
3. Artin 2.2.4 on page 70.
4. Artin 2.2.6 on page 70.
5. Let $B$ be the subset of $\mathrm{GL}_n(\mathbb{R})$ consisting of upper-triangular matrices. Show that $B$ is a subgroup of $\mathrm{GL}_n(\mathbb{R})$.
6. Let $T$ be the subset of $\mathrm{GL}_n(\mathbb{R})$ consisting of diagonal matrices. Show that $T$ is a subgroup of $\mathrm{GL}_n(\mathbb{R})$.
7. Show that the set of matrices

$$H = \left\{ \begin{pmatrix} 1 & x_1 & x_2 & \dots & x_{n-1} & x_n & z \\ 0 & 1 & 0 & 0 & \dots & 0 & y_1 \\ 0 & 0 & 1 & 0 & \dots & 0 & y_2 \\ & & & \ddots & & & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & y_n \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix} \mid x_1, \dots, x_n, y_1, \dots, y_n, z \in \mathbb{R} \right\}$$

   forms a subgroup of $\mathrm{GL}_{n+2}(\mathbb{R})$. It is called the Heisenberg group.

8. For a matrix $A \in M_{n\times n}(\mathbb{R})$, let $A^t$ be the transpose matrix, so that the $ij$-entry of $A^t$ is the $ji$ entry of $A$. Prove that if $A \in \mathrm{GL}_n(\mathbb{R})$, then $(A^{-1})^t = (A^t)^{-1}$. [Hint: recall that for matrices $A$ and $B$ in $M_{n\times n}(\mathbb{R})$, $(AB)^t = B^t A^t$.]
9. (This is the Euclidean algorithm) Let $a, b \in \mathbb{Z}_{\geq 1}$ and consider the division with remainder $a = bq + r$, with $0 \leq r < b$.
   (a) Show that $(a, b) = (b, r)$.
   (b) Write $r_{-1} = a$ and $r_0 = b$ and define the sequence $(r_n)$ recursively using division with remainder $r_{n-1} = r_n q_n + r_{n+1}$ with $0 \leq r_{n+1} < r_n$. Show that if $r_n > 0$ and $r_{n+1} = 0$ then $r_n = (a, b)$.
10. (This is explicit Bezout. This seems elaborate but it really is straightforward and I recommend you do it.) Suppose $a, b \in \mathbb{Z}_{n\geq 1}$. We define the sequences $(r_n)$, $(q_n)$, $(u_n)$ and $(v_n)$ recursively as follows: $r_{-1} = a$, $r_0 = b$, and for $n \geq 0$ define $q_{n+1}$ and $r_{n+1}$ using the division with remainder $r_{n-1} = r_n q_{n+1} + r_{n+1}$ with $0 \leq r_{n+1} < r_n$. Also define $u_{-1} = 1$, $v_{-1} = 0$, $u_0 = 0$, $v_0 = 1$ and for $n \geq 0$

$$u_{n+1} = u_{n-1} - q_{n+1} u_n$$
$$v_{n+1} = v_{n-1} - q_{n+1} v_n$$

(a) Show that $r_n = au_n + bv_n$ by induction on $n$.

(b) Show that $(a, b) = au_N + bv_N$ where $N$ is the largest index such that $r_N > 0$. Here you may use the previous exercise whether or not you actually did it.

(c) (Optional) Use this algorithm to find $m$ and $n$ such that $17m + 23n = 1$. [This is how a computer solves Bezout.]