Math 30810 Honors Algebra 3
Homework 3

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Due Thursday, September 15

Do any 8 of the following 10 questions. Artin a.b.c means chapter a, section b, exercise c.

1. Artin 2.4.3
2. Artin 2.4.7
3. Artin 2.6.2
4. For a matrix $A \in M_{n\times n}(\mathbb{C})$ define $e^A = I_n + A + A^2/2! + A^3/3! + \cdots$. You may assume that this expression always converges to a matrix $e^A \in M_{n\times n}(\mathbb{C})$. If $S \in \mathrm{GL}_n(\mathbb{C})$ show that $e^{SAS^{-1}} = Se^AS^{-1}$.
5. In the context of the previous exercise show that if A is upper triangular with $a_1, \dots, a_n$ on the diagonal then e^A is upper triangular with $e^{a_1}, \dots, e^{a_n}$ on the diagonal and conclude that $\det(e^A) = e^{\mathrm{Tr}(A)}$. As an optional exercise show that for any matrix A, $\det(e^A) = e^{\mathrm{Tr}(A)}$ and deduce that e^A is always invertible. [Hint: You may use the following standard fact from linear algebra, that for every matrix A you can find an invertible matrix S such that SAS^{-1} is upper triangular.]
6. Let G be a group with subgroups H and K. Show that $H \cup K$ is a group if and only if one of H and K contains the other.
7. Show that every cyclic group is abelian.
8. Let G be a group and $g \in G$. Show directly that g and g^{-1} have the same order.
9. Prove that every subgroup of a cyclic group is cyclic.
10. Let $n \geq 2$ be an integer. To a permutation $\sigma \in S_n$ attach the matrix $P(\sigma) = (a_{ij})$ such that for every i, $a_{\sigma(i),i} = 1$ and $a_{i,j} = 0$ if $i \neq \sigma(j)$. Show that P is a homomorphism $P : S_n \to \mathrm{GL}_n(\mathbb{R})$.