Math 30810 Honors Algebra 3
Homework 4

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Due Thursday, September 22

Do any 8 of the following questions. Artin a.b.c means chapter a, section b, exercise c.

1. Artin 2.5.5
2. Artin 2.6.7
3. Artin 2.6.9
4. Suppose H is a subgroup of a group G. Show that the definition of normality from class is equivalent to the one from the textbook, i.e., show that if H is a subgroup such that $gHg^{-1} \subset H$ for all $g \in G$, then $gHg^{-1} = H$ for all $g \in G$.
5. Let G be a group. Recall that $\mathrm{End}(G)$ is the set of homomorphisms $f : G \to G$ and $\mathrm{Aut}(G) \subset \mathrm{End}(G)$ is the subset of those homomorphisms which are isomorphisms.
   (a) Show that usual composition of functions yields an associative composition law on $\mathrm{End}(G)$ with identity given by the identity function. (There is something you need to check here!)
   (b) Show that $f \in \mathrm{End}(G)$ has an inverse with respect to the composition law iff $f \in \mathrm{Aut}(G)$ and conclude that $\mathrm{Aut}(G)$ is a group.
6. Let G be a group. Recall from class that if $g \in G$ then the map $\phi_g(x) = gxg^{-1}$ is a homomorphism $\phi_g \in \mathrm{End}(G)$.
   (a) Show that in fact $\phi_g \in \mathrm{Aut}(G)$.
   (b) Show that the map $\Phi : G \to \mathrm{Aut}(G)$ given by $\Phi(g) = \phi_g$ is a group homomorphism.
   (c) (Optional) Show that $\ker \Phi = Z(G)$, the center of the group G.
7. Show that $\mathrm{Inn}(G)$, defined as the set of all inner automorphism $\{\phi_g \mid g \in G\}$, is a normal subgroup of $\mathrm{Aut}(G)$. [Hint: Use the previous problem.]
8. Show that if G is a group of order 4 then either it is cyclic or it is isomorphic to the Klein 4-group V.
9. Show that $\langle(123)\rangle$ is normal in S_3 but the subgroups $\langle(12)\rangle$, $\langle(13)\rangle$ and $\langle(23)\rangle$ are not normal in S_3.
10. Show that in S_n, $(i_1, \ldots, i_k) = (i_1, i_2)(i_2, i_3) \cdots (i_{k-1}, i_k)$ for any cycle $(i_1, \ldots, i_k)$.
11. (This is a useful problem) For $1 \le i, j \le n$ consider the matrix $E_{ij} \in M_{n\times n}(\mathbb{C})$ with 1 in position ij and 0s everywhere else.
    (a) For $i \neq j$ show that $I_n + E_{ij} \in \mathrm{GL}_n(\mathbb{C})$.
    (b) For a general matrix $X \in \mathrm{GL}_n(\mathbb{C})$ compute XE_{ij} and $E_{ij}X$ and show that $Z(\mathrm{GL}_n(\mathbb{C})) = \mathbb{C}^\times I_n$.