Math 30810 Honors Algebra 3
Homework 5

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Due Thursday, September 29

Do any 8 of the following questions. Artin a.b.c means chapter a, section b, exercise c.

1. Let G be a group and $g \in G$. Suppose $g^m = e$ and $g^n = e$ where m and n are coprime integers. Show that $g = e$.
2. Let G be a group.
   (a) Assume that H and K are subgroups and $|H| = |K| = p$ is a prime number. Show that either $H = K$ or $H \cap K = \{e\}$.
   (b) Let G be a group and $H_1, \ldots, H_k$ be distinct subgroups of G. Suppose that each group H_i has order p, a fixed prime number. Show that $H_1 \cup \ldots \cup H_k$ has exactly $(p-1)k+1$ elements.
3. Suppose G is a finite group and p is a prime number such that every element $g \in G - \{e\}$ has order p. Show that $p - 1 \mid |G| - 1$. [Hint: use exercise 2.]
4. Let G be a group and suppose G contains an element of order n. Show that for every divisor $d \mid n$ the group G contains an element of order d. Deduce that if G has order p^n for some prime p, G contains an element of order p.
5. Let $p > q$ be prime numbers such that $q - 1 \nmid p - 1$, and suppose G is a group of order exactly pq. (E.g., G could have order 35.) Show that G contains an element of order p and an element of order q. [Hint: you may find exercises 3 and 4 useful.]
6. (I encourage you to do this problem) Let $G = \mathrm{GL}_2(\mathbb{R})$ and H the subgroup of upper triangular matrices. Show that a complete set of representatives of G/H is given by the matrices

$$\left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\} \sqcup \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

   (The proper interpretation of this is that the first set of matrices represents the real line and the antidiagonal matrix represents the "point at infinity", the quotient G/H being the projective line. This is important in representation theory.)
7. Artin 2.8.8 on page 73.
8. Artin 2.8.10 on page 73.
9. Artin 2.9.3 on page 73.
10. Artin 2.8.6 on page 73.
11. (This was one is fun and jocular) Artin 2.M.16 on page 77. Artin says he learned of this from a paper of Mestre, Schoof, Washington and Zagier. The paper starts with the "motto": *Ah! La recherche. Du temps perdu.*