Math 30810 Honors Algebra 3
Homework 6

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Due Thursday, October 6

Do any 8 of the following questions. Artin a.b.c means chapter a, section b, exercise c.

1. Explicit Chinese Remainder Theorem.

   (a) Let m and n be coprime integers and let u and v be integers such that $mu+nv=1$ (from Bézout's relation). Show that the system of equations

   $$\begin{cases} x \equiv a \pmod{m} \\ x \equiv b \pmod{n} \end{cases}$$

   has the unique solution $x \equiv anv + bmu \pmod{mn}$.

   (b) Compute

   $$12^{34^{56^{78}}} \pmod{90}$$

   [Hint: Use the Chinese Remainder Theorem.] (A bit on notation: the exponent of 56 is 78, the exponent of 34 is 56^{78}, the exponent of 12 is $34^{56^{78}}$. In particular, this is NOT $((12^{34})^{56})^{78}$.)

2. Artin 2.9.5 on page 73.

3. Let p be a prime integer. Show that $(p-1)! \equiv -1 \pmod{p}$. [Hint: There are two ways to do this. Either (a) decompose the polynomial $X^{p-1}-1 \mod p$ into linear factors or (b) interpret $(p-1)!$ as a product of elements in $(\mathbb{Z}/p\mathbb{Z})^\times$.]

4. Artin 2.12.1 on page 74.

5. Artin 2.12.2 on page 75.

6. Let n be a positive integer and $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in (\mathbb{Z}/n\mathbb{Z})^\times, b \in \mathbb{Z}/n\mathbb{Z} \right\}$ and $H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{Z}/n\mathbb{Z} \right\}$. Show that G is a group under usual matrix multiplication and H is a normal subgroup of G. (The group G will be a Galois group next semester, so this is a useful problem.)

7. Let G be a finite group and $g \in G$ not the identity. Show that g has order m if and only if the following two conditions are satisfied:

   (a) $g^m = e$ and

   (b) for every prime divisor $p \mid m$, $g^{m/p} \neq e$.

8. (We will use this exercise in class so try to do it) Suppose G is an abelian group containing an element g of order p^{k+1} where p is a prime and an element h of order $p^k m$ where $p \nmid m$. Show that $p^{k+1}m \mid \operatorname{ord}(gh)$.

9-10 (Counts as two problems) Consider the permutations $a_1 = (12)(34)$, $a_2 = (13)(24)$ and $a_3 = (14)(23)$ in S_4. Let $X = \{a_1, a_2, a_3\}$.

(a) If $\sigma \in S_4$ show that the inner automorphism $\phi_\sigma(x) = \sigma x \sigma^{-1}$ of S_4 yields a bijective function $\phi_\sigma|_X : X \to X$. I.e., you need to check that ϕ_σ takes X to X and that it is a bijection on X.

(b) For $\sigma \in S_4$ define the permutation $c_\sigma \in S_3$ such that $\phi_\sigma(a_1) = a_{c_\sigma(1)}$, $\phi_\sigma(a_2) = a_{c_\sigma(2)}$ and $\phi_\sigma(a_3) = a_{c_\sigma(3)}$. Show that the map $q : S_4 \to S_3$ defined by $q(\sigma) = c_\sigma$ is a group homomorphism.

(c) Show that q is surjective. [Hint: It suffices to show that the image of q contains a transposition and a 3-cycle as we showed in class that S_3 is generated by two such elements.]

(d) Show that $\ker q = X \cup \{e\}$. [Hint: show that $\ker q$ contains $X \cup \{e\}$ and then use the first isomorphism theorem.]

(e) Conclude that $\ker q$ is a normal subgroup of order 4 of the alternating group A_4. (If $n = 3$ or $n \geq 5$ the only normal subgroups of A_n are the trivial subgroup and A_n itself, so A_4 is exceptional.)