Math 30810 Honors Algebra 3
Homework 8

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Due at noon on Thursday, October 27

Do 8 of the following questions. Some questions are obligatory. Artin a.b.c means chapter a, section b, exercise c. You may use any problem to solve any other problem.

1. (You have to do this problem) A **short exact sequence** of groups is a sequence of group homomorphisms
$$1 \to N \xrightarrow{f} G \xrightarrow{g} K \to 1$$
such that $f : N \to G$ is injective, $g : G \to K$ is surjective, and $\operatorname{Im} f = \ker g$. A **section** of such an exact sequence is defined to be a group homomorphism $s : K \to G$ such that $g \circ s = \mathrm{id}_K$.
   (a) Show that in the short exact sequence above $N \cong f(N) \lhd G$ and $G/f(N) \cong K$.
   (b) Suppose that the exact sequence above admits a section $s : K \to G$. Show that for every $x \in G$ one can find $n \in N$ such that $x = f(n)s(g(x))$ and deduce that $G \cong N \rtimes K$ is a semidirect product.
   (c) (Extra credit) Show that if $G \cong N \rtimes K$ then one can find an exact sequence $1 \to N \to G \to K \to 1$ that admits a section $s : K \to G$.

2. Recall from class[1] that the matrices $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generate $\mathrm{SL}(2, \mathbb{Z})$.
   (a) Show that $\mathrm{SL}(2, \mathbb{Z}) = \langle S, ST \rangle$ and that the two generators S and ST have orders 4 respectively 6.
   (b) (Do either this or the next part) Show that the only homomorphism $f : \mathrm{SL}(2, \mathbb{Z}) \to \mathbb{Z}/7\mathbb{Z}$ is the trivial homomorphism. [Hint: It's enough to see where the generators go.]
   (c) (Do either this or the previous part) Show that every homomorphism $f : \mathrm{SL}(2, \mathbb{Z}) \to \mathbb{C}^\times$ has image $\operatorname{Im} f \subset \mu_{12} = \{z \in \mathbb{C}^\times \mid z^{12} = 1\}$.

3. Let $\zeta = e^{2\pi i/3}$, $x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$. Let $G = \langle x, y \rangle \subset \mathrm{GL}(2, \mathbb{C})$ be the subgroups generated by x and y.
   (a) Show that x has order 4, y has order 3 and $xy = y^2 x$.
   (b) Show that G has order 12 with $G = \{y^b x^a \mid 0 \le a < 4, 0 \le b < 3\}$.
   (c) Show that $G \cong \mathbb{Z}/3\mathbb{Z} \rtimes_\phi \mathbb{Z}/4\mathbb{Z}$ for some $\phi : \mathbb{Z}/4\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/3\mathbb{Z})$. [Hint: Use the criterion for when a group is a semidirect product.]

4. Consider the homomorphism $\phi : S_3 \to \mathrm{Inn}(S_3) = \mathrm{Aut}(S_3)$ defined by $\phi_g(x) = gxg^{-1}$. Consider the groups $G_0 = S_3 \rtimes_\phi S_3$, $G_1 = S_3 \rtimes_\phi A_3$ and $G_2 = A_3 \rtimes_\phi A_3$.

[1] Actually in class I showed this with the inverse of the matrix S, but this is the more standard version of S

(a) Show that $G_2 \cong (\mathbb{Z}/3\mathbb{Z})^2$ [Hint: $A_3 \cong \mathbb{Z}/3\mathbb{Z}$],
(b) Show that $G_2 \lhd G_1$ with $G_1/G_2 \cong \mathbb{Z}/2\mathbb{Z}$,
(c) Show that $G_1 \lhd G_0$ with $G_0/G_1 \cong \mathbb{Z}/2\mathbb{Z}$.

[Hint: To show normality you can use a criterion from a previous problem set. No need to do any conjugation.]

5. Show that S_n is generated by the transpositions $(12), (23), \dots, (n-1,n)$. [Hint: $(23)(12)(23) = (13)$. Recall that in class we showed that S_n is generated by all transpositions.]

6. (a) Show that $(12\dots n)(i,i+1)(12\dots n)^{-1} = (i+1,i+2)$ for $i+2 \leq n$.
   (b) Show that $(12\dots n)^k(12)(12\dots n)^{-k} = (k+1,k+2)$ for $k+2 \leq n$.
   (c) Deduce that S_n is generated by (12) and $(12\dots n)$. [Hint: Use the previous problem.]

7. Consider $\mathbb{Q}$ as a group with respect to $+$. Show that every finitely generated subgroup of $\mathbb{Q}$ is of the form $q\mathbb{Z}$ for some rational $q \in \mathbb{Q}$. In other words, every finitely generated subgroup is cyclic so generated by one single element.

8. Let $G = \left\langle \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle \subset \mathrm{GL}(2,\mathbb{R})$ and let $H < G$ be the subgroup of G consisting of those matrices with 1-s on the diagonal. Show that H is not finitely generated, i.e., there don't exist finitely many matrices $g_1, \dots, g_n \in H$ such that $H = \langle g_1, g_2, \dots, g_n \rangle$. [Hint: Show that H is a subgroup of $\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{Q} \right\} \cong \mathbb{Q}$, but H contains matrices where the upper right corner is a rational with arbitrarily large powers of 2 in the denominator. You may use Exercise 7.]

   Remark: The point of this exercise is to show that subgroups of finitely generated groups are not necessarily finitely generated.

9. Let $n \geq 3$. Consider the dihedral group $D_{2n} = (\mathbb{Z}/n\mathbb{Z}) \rtimes_\phi (\mathbb{Z}/2\mathbb{Z})$ where $\phi : \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$ is defined by $\phi_0 = \mathrm{id}_{\mathbb{Z}/n\mathbb{Z}}$ and $\phi_1(x) = -x$. Recall from class that if $F = (1,0)$ and $R = (0,1)$ (the first coordinate in $\mathbb{Z}/2\mathbb{Z}$ and the second coordinate in $\mathbb{Z}/n\mathbb{Z}$) then F has order 2, R has order n and $FRF = R^{-1}$, and $D_{2n} = \langle F, R \rangle$. Suppose $a, b \in \mathbb{Z}/n\mathbb{Z}$. Show that R^a and FR^b generate D_{2n} (i.e., $D_{2n} = \langle R^a, FR^b \rangle$) if and only if $a \in (\mathbb{Z}/n\mathbb{Z})^\times$. [Hint: Show that in an arbitrary product of R^a-s and FR^b-s and their inverses you can collect all the F-s on the left side.]

10-11 (This counts as two problems) Let $n \geq 3$ be an odd number. Consider the group homomorphism $\phi : (\mathbb{Z}/n\mathbb{Z})^\times \to \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$ given by $\phi_a(x) = ax$. Recall that the dihedral group $D_{2n} = \{F^u R^v \mid 0 \leq u \leq 1, 0 \leq v < n\}$.

   (a) For $a \in (\mathbb{Z}/n\mathbb{Z})^\times$ and $b \in \mathbb{Z}/n\mathbb{Z}$ define $\Psi_{a,b}(F^u R^v) := (FR^b)^u (R^a)^v$. Show that $\Psi_{a,b} \in \mathrm{Aut}(D_{2n})$. [Hint: Use the previous problem.]
   (b) Show that $\Psi : (\mathbb{Z}/n\mathbb{Z}) \rtimes_\phi (\mathbb{Z}/n\mathbb{Z})^\times \to \mathrm{Aut}(D_{2n})$ is an injective group homomorphism.
   (c) Show that Ψ is surjective, i.e., that every automorphism of D_{2n} is of the form $\Psi_{a,b}$ for some a and b and conclude that
   $$\mathrm{Aut}(D_{2n}) \cong (\mathbb{Z}/n\mathbb{Z}) \rtimes_\phi (\mathbb{Z}/n\mathbb{Z})^\times$$
   [Hint: Use part (a).]
   (d) (Extra credit) For a group G the group of outer automorphisms is defined as $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$, a group since $\mathrm{Inn}(G) \lhd \mathrm{Aut}(G)$ from a previous homework. Show that $\mathrm{Out}(D_{2n}) \cong (\mathbb{Z}/n\mathbb{Z})^\times/\{\pm 1\}$.

   Remark: If G is a group and $\phi : \mathrm{Aut}(G) \to \mathrm{Aut}(G)$ is the identity homomorphism then the semidirect product $G \rtimes_\phi \mathrm{Aut}(G)$ is called the holomorph of G. The point of this exercise was to show that $\mathrm{Aut}(D_{2n})$ when $n \geq 3$ was odd was the holomorph of $\mathbb{Z}/n\mathbb{Z}$. In fact this is true for all n.