Math 30810 Honors Algebra 3
Homework 9

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Due at noon on Thursday, November 3

Do 8 of the following questions. Some questions are obligatory. Artin a.b.c means chapter a, section b, exercise c. You may use any problem to solve any other problem.

1-2 (Counts as 2 problems) Let G be a finite group and H a subgroup of G. Denote by $S_{G/H}$ the group of permutations of the finite set G/H. If G/H has k elements then $S_{G/H} \cong S_k$, the group operation on both sides being composition of permutations.

(a) Show that if $h \in H$ then the map $f_g : G/H \to G/H$ defined by $f(xH) = gxH$ is a bijection, in other words $f_g \in S_{G/H}$.

(b) Show that the map $\Phi : G \to S_{G/H}$ given by $\Phi(g) = f_g$ is a group homomorphism with $\ker \Phi \subset H$.

(c) Suppose the index $[G : H] = p$ is the least prime divisor of the order $|G|$. Show that $|G/\ker \Phi| = p$ and deduce that H is normal in G. (This is a generalization of a previous homework problem that stated that index 2 subgroups are normal. Indeed 2 is the least prime divisor of every even order.) [Hint: what is the cardinality of $S_{G/H}$?]

3. For a matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R})$ and $z \in \mathbb{C}$ define, if possible, $g \cdot z = \dfrac{az + b}{cz + d}$.

(a) Show that $\mathrm{Im}(g \cdot z) = \dfrac{\det(g)\,\mathrm{Im}(z)}{|cz + d|^2}$.

(b) Show that $g \cdot z$ defined an action of the subgroup $\mathrm{GL}(2, \mathbb{R})^+$ of matrices with positive determinant on the set $\mathcal{H} = \{z \in \mathbb{C} | \,\mathrm{Im}\, z > 0\}$.

(c) Compute the stabilizers $\mathrm{Stab}(i)$ and $\mathrm{Stab}(\zeta_3)$.

(d) (Optional) Show that this action is transitive, i.e., all of $\mathcal{H}$ is one orbit.

4. Recall from class that the group $\mathrm{GL}(2, \mathbb{Z})$ acts via usual matrix multiplication on the left on $\mathbb{Z}^2$.

(a) Suppose $u, v \neq 0$ are two integers. Show that there exist two integers w, t such that $\begin{pmatrix} u & w \\ v & t \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z})$ if and only if $(u, v) = 1$.

(b) If $d \in \mathbb{Z}_{\geq 1}$ show that the orbit of $\begin{pmatrix} d \\ 0 \end{pmatrix}$ consists of vectors $\begin{pmatrix} a \\ b \end{pmatrix}$ with gcd $(a, b) = d$. [Hint: Use part (a).]

(c) Show that the set $S = \{\begin{pmatrix} d \\ 0 \end{pmatrix} | d \in \mathbb{Z}_{\geq 0}\}$ parametrizes the orbits of $\mathrm{GL}(2, \mathbb{Z})$ acting on $\mathbb{Z}^2$, i.e., every orbit contains a unique element from the set S. (You can think of S as a complete set of representatives for the orbits.) [Hint: Use part (b).]

5. Show that the matrices $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $A^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ are conjugate in $\mathrm{GL}(2, \mathbb{R})$ but not conjugate in $\mathrm{SL}(2, \mathbb{R})$.
6. Artin 6.7.3 on page 190.
7. Artin 6.7.7 on page 191.
8. Artin 6.8.1 on page 191.
9. Artin 6.M.7 on page 194. (Careful: what Artin calls D_3 in part (a) we called D_6, it is the dihedral group with 6 elements, isomorphic to S_3.)
10. Artin 7.2.5 on page 221.