Math 30810 Honors Algebra 3
Homework 9

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Due at noon on Thursday, November 3

Do 8 of the following questions. Some questions are obligatory. Artin a.b.c means chapter a, section b, exercise c. You may use any problem to solve any other problem.

1-2 (Counts as 2 problems) Let G be a finite group and H a subgroup of G. Denote by $S_{G/H}$ the group of permutations of the finite set G/H. If G/H has k elements then $S_{G/H} \cong S_k$, the group operation on both sides being composition of permutations.

(a) Show that if $h \in H$ then the map $f_g : G/H \to G/H$ defined by $f(xH) = gxH$ is a bijection, in other words $f_g \in S_{G/H}$.

(b) Show that the map $\Phi : G \to S_{G/H}$ given by $\Phi(g) = f_g$ is a group homomorphism with $\ker \Phi \subset H$.

(c) Suppose the index $[G : H] = p$ is the least prime divisor of the order $|G|$. Show that $|G/\ker \Phi| = p$ and deduce that H is normal in G. (This is a generalization of a previous homework problem that stated that index 2 subgroups are normal. Indeed 2 is the least prime divisor of every even order.) [Hint: what is the cardinality of $S_{G/H}$?]

Proof. (a): If $f_g(xH) = f_g(yH)$ then $gxH = gyH$ so $xH = yH$ so f_g is injective. Multiplication by g is surjective on G and so f_g is also surjective. Thus f_g is a permutation of G/H.

(b): If $g, h \in G$, $f_g \circ f_h(xH) = ghxH = f_{gh}(xH)$ so $\Phi : G \to S_{G/H}$ is a group homomorphism. If $g \in \ker \Phi$ then $f_g = \mathrm{id}$ so $f_g(H) = H$. Thus $gH = H$ so $g \in H$ and we deduce $\ker f \subset H$.

(c): $G/\ker \Phi \cong \operatorname{Im} \Phi$ which is a subgroup of $S_{G/H}$. By Lagrange $|G/\ker \Phi| \mid |S_{G/H}| = p!$. But $|G/\ker \Phi| \mid |G|$ so $|G/\ker \Phi| \mid (p!, |G|) = p$. Thus $\ker \Phi \subset H \subset G$ with $[G : H] = [G : \ker \Phi]$ so $H = \ker f$ which is then normal in G. □

3. For a matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R})$ and $z \in \mathbb{C}$ define, if possible, $g \cdot z = \dfrac{az + b}{cz + d}$.

(a) Show that $\operatorname{Im}(g \cdot z) = \dfrac{\det(g) \operatorname{Im}(z)}{|cz + d|^2}$.

(b) Show that $g \cdot z$ defined an action of the subgroup $\mathrm{GL}(2, \mathbb{R})^+$ of matrices with positive determinant on the set $\mathcal{H} = \{z \in \mathbb{C} | \operatorname{Im} z > 0\}$.

(c) Compute the stabilizers $\mathrm{Stab}(i)$ and $\mathrm{Stab}(\zeta_3)$.

(d) (Optional) Show that this action is transitive, i.e., all of $\mathcal{H}$ is one orbit.

Proof. (a) Note that $\operatorname{Im} z = (2i)^{-1}(z + \overline{z})$ and so

$$\begin{aligned}\operatorname{Im} g \cdot z &= (2i)^{-1}(g \cdot z + \overline{g \cdot z}) \\ &= (2i)^{-1}(\frac{az+b}{cz+d} + \frac{a\overline{z}+b}{c\overline{z}+d}) \\ &= (2i)^{-1}\frac{(ad-bc)(z-\overline{z})}{(cz+d)(c\overline{z}+d)} \\ &= \frac{\det g \operatorname{Im} z}{|cz+d|^2}\end{aligned}$$

(b): If $\det g > 0$ and $\operatorname{Im} z > 0$ then by (a) $\operatorname{Im} g \cdot z > 0$ so $\mathrm{GL}(2,\mathbb{R})^+$ preserves $\mathcal{H}$. Need to check action, i.e., that $(gh) \cdot z = g \cdot (h \cdot z)$. But

$$\begin{pmatrix} m & n \\ p & q \end{pmatrix} \cdot (\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z) = \frac{m\frac{az+b}{cz+d} + n}{p\frac{az+b}{cz+d} + q} = \frac{(am+cn)z + mb + dn}{(ap+cq)z + bp + dq} = \begin{pmatrix} m & n \\ p & q \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z$$

(c): We seek $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\frac{ai+b}{ci+d} = i$, i.e., $a = d$ and $c = -b$ so $\mathrm{Stab}(i) = \{\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\}$. We now seek a, b, c, d such that $\frac{a\zeta_3+b}{c\zeta_3+d} = \zeta_3$ which multiplies out to $a\zeta_3 + b = c\zeta_3^2 + d\zeta_3 = -c + (d-c)\zeta_3$. Thus $a = d - c$ and $b = -c$ so $\mathrm{Stab}(\zeta_3) = \{\begin{pmatrix} a & b \\ -b & a-b \end{pmatrix}\}$.

(d): If $z = x + iy$ then

$$z = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \cdot i$$

and $\operatorname{Im} z = \det \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} > 0$. Thus every z is in the orbit of i.

□

4. Recall from class that the group $\mathrm{GL}(2,\mathbb{Z})$ acts via usual matrix multiplication on the left on $\mathbb{Z}^2$.

   (a) Suppose $u, v \neq 0$ are two integers. Show that there exist two integers w, t such that $\begin{pmatrix} u & w \\ v & t \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z})$ if and only if $(u, v) = 1$.

   (b) If $d \in \mathbb{Z}_{\geq 1}$ show that the orbit of $\begin{pmatrix} d \\ 0 \end{pmatrix}$ consists of vectors $\begin{pmatrix} a \\ b \end{pmatrix}$ with gcd $(a, b) = d$. [Hint: Use part (a).]

   (c) Show that the set $S = \{\begin{pmatrix} d \\ 0 \end{pmatrix} | d \in \mathbb{Z}_{\geq 0}\}$ parametrizes the orbits of $\mathrm{GL}(2,\mathbb{Z})$ acting on $\mathbb{Z}^2$, i.e., every orbit contains a unique element from the set S. (You can think of S as a complete set of representatives for the orbits.) [Hint: Use part (b).]

Proof. (a): If $u = du'$ and $v = dv'$ for $d = (u, v)$ it follows that $\det \begin{pmatrix} u & w \\ v & t \end{pmatrix} = d \det \begin{pmatrix} u' & w \\ v' & t \end{pmatrix}$ and so the LHS has determinant ± 1 implies that $d = \pm 1$ so u and v are coprime. If u and v are coprime then Bezout implies we can find w, t such that $ut - vw = 1$ and so det of the LHS is 1 as desired.

(b): Suppose $(a, b) = d$ so $a = a'd$ and $b = b'd$ with $(a', b') = 1$. Then there exist (Bezout) integers p, q such that $a'p + b'q = 1$. Let $g = \begin{pmatrix} p & q \\ -b' & -a' \end{pmatrix}$. Then $g \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} d \\ 0 \end{pmatrix}$ and $\det g = 1$. Thus $\begin{pmatrix} a \\ b \end{pmatrix}$ is in the orbit of $\begin{pmatrix} d \\ 0 \end{pmatrix}$. Reciprocally, suppose $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \cdot \begin{pmatrix} d \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$. Then $pd = a$ and $rd = b$ and so $d \mid a, b$. But $(p, r) = 1$ from part (a) so $(a, b) = d$.

(c): If $v = \begin{pmatrix} a \\ b \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ then v is in the orbit of $\begin{pmatrix} (a, b) \\ 0 \end{pmatrix}$. The zero vector is its own orbit. Clearly $\begin{pmatrix} a \\ 0 \end{pmatrix}$ and $\begin{pmatrix} b \\ 0 \end{pmatrix}$ with $a, b \geq 0$ are in the same orbit iff $a = (b, 0) = b$. □

5. Consider the matrices $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ in $\mathrm{SL}(2, \mathbb{R})$. Show that A and B are conjugate in $\mathrm{GL}(2, \mathbb{R})$ but not conjugate in $\mathrm{SL}(2, \mathbb{R})$.

Proof. If $S = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ then $SAS^{-1} = B$ and since $S \in \mathrm{GL}(2, \mathbb{R})$ the matrices are conjugate in the larger group. Let's show A and B are not $\mathrm{SL}(2, \mathbb{R})$ conjugate. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$ then $\det g = ad - bc = 1$ and so $g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. We compute

$$gAg^{-1} = \begin{pmatrix} 1 - ac & a^2 \\ -c^2 & 1 + ac \end{pmatrix}$$

and since the top right corner of this conjugate is always $a^2 \geq 0$, no $\mathrm{SL}(2, \mathbb{R})$ conjugate of A can ever be B whose top right corner is -1. □

6. Artin 6.7.3 on page 190.

Proof. Write $U = \{u_1, u_2, u_3\}$ and $V = \{v_1, v_2, v_3\}$. In both cases S_3 acts transitively on U so $O_{u_i} = U$. From class we know that $S_3/\operatorname{Stab}(x)$ is in bijection with O_x and so we deduce that $\operatorname{Stab}(u_i)$ is an order 2 subgroup of S_3. Again from class we know that $\operatorname{Stab}(\sigma x) = \sigma \operatorname{Stab}(x) \sigma^{-1}$ and so by varying σ we deduce that every order 2 subgroup of S_3 appears as a stabilizer for the action of S_3 on U as the order 2 subgroups of S_3 form a conjugacy class in S_3 (again from class). By reordering we can assume that u_i has stabilizer $\langle (jk) \rangle$ where $i, j, k = 1, 2, 3$ reordered.

(a): Applying the previous paragraph to V we may assume that v_i has stabilizer $\langle (jk) \rangle$. By definition of stabilizers we see that $\operatorname{Stab}((u_i, v_j)) = \operatorname{Stab}(u_i) \cap \operatorname{Stab}(v_j)$ and therefore $\operatorname{Stab}((u_i, v_j)) = 1$ unless $i = j$ in which case $\operatorname{Stab}((u_i, v_i)) = \langle (jk) \rangle$. Again $O_{(u_i, v_j)}$ is in bijection with $S_3/\operatorname{Stab}((u_i, v_j))$ and the orbits disjointly cover $U \times V$ so by inspection we deduce the orbits of S_3 on $U \times V$ are

$$O_{(u_1, v_1)} = \{(u_i, v_i) \mid i = 1, 2, 3\}$$
$$O_{(u_1, v_2)} = \{(u_i, v_j) \mid i \neq j\}$$

(b): Clearly $\operatorname{Stab}(v_1) = S_3$ by assumption. Moreover, $\operatorname{Stab}(v_2)$ has order 3 so it must be A_3. Since $g(u_i, v_1) = (gu_i, v_1)$ it follows that $U \times \{v_1\}$ is an orbit for S_3. If $r \neq 1$, as in part (a) we get $\operatorname{Stab}((u_i, v_r)) = \operatorname{Stab}(u_i) \cap \operatorname{Stab}(v_r) = \langle (jk) \rangle \cap A_3 = 1$ and so the orbit of (u_i, v_r) has cardinality 6. We deduce that the orbits are

$$O_{(u_1, v_1)} = \{(u_i, v_1) \mid i = 1, 2, 3\}$$
$$O_{(u_1, v_2)} = \{(u_i, v_j) \mid j \neq 1\}$$

□

7. Artin 6.7.7 on page 191.

Proof. (a): First, clearly $O_0 = \{0\}$.

If $a_1 \neq 0$ then note that

$$\begin{pmatrix} a_1 & 0 & \dots & 0 \\ a_2 & 1 & 0 & \dots \\ \vdots & & \ddots & \\ a_n & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

so $v = (a_1, \dots, a_n) \in O_{e_1}$. Otherwise, suppose $a_k \neq 0$. Consider the matrix $S = (s_{ij})$ which has 1-s on the diagonal in position (i,i) if $i \neq 1, k$, $s_{1,k} = 1$, $s_{k,1} = 1$ and S has 0-s everywhere else. Then the above shows that $v' = (a_k, a_2, \dots, a_{k-1}, a_1, a_{k+1}, \dots) \in O_{e_1}$ and simply note that $Sv' = v$ so $v \in O_{e_1}$ as well.

(b): $\text{Stab}(e_1) = \{g \in \text{GL}_n(\mathbb{R}) \mid ge_1 = e_1\} = \{\begin{pmatrix} 1 & *_{1 \times n-1} \\ O_{n-1 \times 1} & *_{n-1 \times n-1} \end{pmatrix}\}$. □

8. Artin 6.8.1 on page 191.

Proof. Indeed $P * (Q * A) = P * (QAQ^t) = PQAQ^tP^t = PQA(PQ)^t = PQ * A$ and $I_n * A = A$ so this is an action. □

9. Artin 6.M.7 on page 194.

Proof. (a): Recall that $D_6 = \{1, R, R^2, F, FR, FR^2\}$ where R rotates the equilateral triangle $1, \zeta_3, \zeta_3^2$ and F flips it. The table is

	1	ζ_3	ζ_3^2
1	T	T	T
R	F	F	F
R^2	F	F	F
F	T	F	F
FR	F	T	F
FR^2	F	F	T

(b): We need to show that $\sum_{s \in S} |\text{Stab}_G(s)| = \sum_{g \in G} |\{s \in S \mid gs = s\}|$. Note that $|S^g$ is the number of trues on the row corresponding to g in the table. Also, $|\text{Stab}_G(s)|$ is the number of trues on the column corresponding to s in the table. Now the LHS and the RHS count in different ways the total number of trues in the table.

(c): We know that $|G| = |\text{Stab}_G(s)||O_s|$ so now we compute

$$\begin{aligned} \sum_{g \in G} |S^g| &= \sum_{s \in S} |\text{Stab}_G(s)| \\ &= \sum_{s \in S} |G|/|O_s| \\ &= \sum_{\text{orbits} O} \sum_{s \in O} |G|/|O| \\ &= \sum_{O} |G|/|O| \times |O| \\ &= |G| \times \text{number of orbits} \end{aligned}$$

as desired. □

10. Artin 7.2.5 on page 221.

Proof. Write $m(a,b) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. Let's conjugate

$$m(a,b)m(x,y)m(a,b)^{-1} = m(x, ay + (1-x)b)$$

We can vary $a \in (0, \infty)$ and $b \in \mathbb{R}$. If $x \neq 1$ it follows that we can vary b to get all matrices of the form $m(x, *)$ and so when $x \neq 1$ we get $\{m(x,z) \mid z \in \mathbb{R}\}$ are equivalence classes. When $x = 1$ we get that $m(a,b)m(1,y)m(a,b)^{-1} = m(1, ay)$ with $a > 0$. If $y \neq 0$ we can get all matrices of the form $m(1,z)$ where z and y have the same sign. Thus we get 3 more orbits: $\{I_2\}$, $\{m(1,z) \mid z > 0\}$ and $\{m(1,z) \mid z < 0\}$. □