# Math 30810 Honors Algebra 3
# Homework 10

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Due at noon on Thursday, November 10

**Do 8 of the following questions. Some questions are obligatory. Artin a.b.c means chapter a, section b, exercise c. You may use any problem to solve any other problem.**

1. Let $A, B \in M_{n\times n}(\mathbb{R})$ and suppose that there exists a complex matrix $S \in \mathrm{GL}(n,\mathbb{C})$ such that $A = SBS^{-1}$. Write $S = X + iY$ for two matrices $X, Y \in M_{n\times n}(\mathbb{R})$.
   (a) Show that $AX = XB$ and $AY = YB$.
   (b) Show that for some real number $r$ the matrix $T = X + rY$ is in $\mathrm{GL}(n,\mathbb{R})$ and $A = TBT^{-1}$.

   (The point of this problem is to show that if two real matrices are conjugate over $\mathbb{C}$ they are also conjugate over $\mathbb{R}$.)

2. (a) Show that $(123)$ and $(132)$ are not conjugate in $A_3$ or $A_4$.
   (b) (Do this or the next part) Show that if $n \geq 5$ is odd then $(12\ldots n)$ and $(12\ldots n, n-1)$ are not conjugate in $A_n$.
   (c) (Do this or the previous part) Show that if $n \geq 6$ is even then $(12\ldots n-1)$ and $(12\ldots n-2, n)$ are not conjugate in $A_n$.

3. Let $G$ be a group. If $g, h \in G$ are two conjugate elements show that there is a bijection between $\{x \in G \mid g = xhx^{-1}\}$ and $\mathrm{Stab}_G(h)$.

4. Let $n \geq 5$ and $H$ a subgroup of $S_n$. Assume that $H$ is not $A_n$ or $S_n$. Show that $[S_n : H] \geq n$. [Hint: As in Problem 1-2 on homework 9 look at the homomorphism $S_n \to S_{S_n/H}$.]

5. Let $p$ be a prime and $G$ a nonabelian group of order $p^3$. Show that $[G, G] = Z(G)$.

6. Let $n \geq 3$ be odd. Find all conjugacy classes in the dihedral group $D_{2n}$.

7. (a) Show that $\mathrm{PSL}(2,\mathbb{F}_3) := \mathrm{SL}(2,\mathbb{F}_3)/\{\pm I_2\}$ has order 12.
   (b) Show that in $\mathrm{PSL}(2,\mathbb{F}_3)$, $x = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ has order 3, $y = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ and $z = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ have order 2 and commute, and $\mathrm{PSL}(2,\mathbb{F}_3) = \langle x, y, z\rangle$.
   (c) Show that $\mathrm{PSL}(2,\mathbb{F}_3) \cong (\mathbb{Z}/2\mathbb{Z})^2 \rtimes \mathbb{Z}/3\mathbb{Z}$ and conclude that $\mathrm{PSL}(2,\mathbb{F}_3) \cong A_4$. [Hint: Show that $N = \langle y, z\rangle$ is normal in $\mathrm{PSL}(2,\mathbb{F}_3)$. Recall that $\mathrm{GL}(2,\mathbb{Z}/2\mathbb{Z}) \cong S_3$ from a previous homework and show that $A_4$ is a similar semidirect product that must be isomorphic to this one.]

8. (a) Suppose $G$ is a group and $g, h \in G$. Show that $gh$ and $hg$ are conjugate.
   (b) A permutation $\sigma \in S_3$ is said to be good if for every group $G$ and every elements $g_1, g_2, g_3 \in G$, the two products $g_1g_2g_3$ and $g_{\sigma(1)}g_{\sigma(2)}g_{\sigma(3)}$ are conjugate in $G$. Show that $\sigma$ is good if and only if $\sigma \in \langle(123)\rangle$. [Hint: conjugate matrices have the same trace.]

9. Artin 7.3.1 on page 222.

10. Artin 7.5.11 (a) on page 223.