Math 30810 Honors Algebra 3
Homework 10

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Due at noon on Thursday, November 10

Do 8 of the following questions. Some questions are obligatory. Artin a.b.c means chapter a, section b, exercise c. You may use any problem to solve any other problem.

1. Let $A, B \in M_{n\times n}(\mathbb{R})$ and suppose that there exists a complex matrix $S \in \mathrm{GL}(n, \mathbb{C})$ such that $A = SBS^{-1}$. Write $S = X + iY$ for two matrices $X, Y \in M_{n\times n}(\mathbb{R})$.

   (a) Show that $AX = XB$ and $AY = YB$.

   (b) Show that for some real number r the matrix $T = X + rY$ is in $\mathrm{GL}(n, \mathbb{R})$ and $A = TBT^{-1}$.

   (The point of this problem is to show that if two real matrices are conjugate over $\mathbb{C}$ they are also conjugate over $\mathbb{R}$.)

   Proof. (a): Since $AS = SB$ it follows that $AX + iAY = XB + iYB$. As A, B, X, Y are real matrices we conclude that $AX = XB$ and $AY = YB$.

   (b): From (a) if $T = A + rB$ then $AT = TB$. Look at $\det T = \det(A + rB)$ which is a polynomial in r, call if $P(r) \in \mathbb{R}[X]$. Since $P(i) = \det S \neq 0$ it follows that P is not the zero polynomial and therefore it has finitely many real roots. Let $r \in \mathbb{R}$ be such that $P(r) \neq 0$ and then T is real, invertible and $A = TBT^{-1}$. □

2. (a) Show that (123) and (132) are not conjugate in A_3 or A_4.

   (b) (Do this or the next part) Show that if $n \geq 5$ is odd then $(12\dots n)$ and $(12\dots n, n-1)$ are not conjugate in A_n.

   (c) (Do this or the previous part) Show that if $n \geq 6$ is even then $(12\dots n-1)$ and $(12\dots n-2, n)$ are not conjugate in A_n.

   Proof. We start with a claim: suppose $\sigma \in S_n$ and we know that σ permutes $\{1, 2, \dots, k\}$ and there exists a permutation $\tau \in S_k$ such that the cycles $(\sigma(1), \dots, \sigma(k)) = (\tau(1), \dots, \tau(k))$ are equal. Then σ restricted to $\{1, 2, \dots, k\}$ is the permutation $\tau \cdot (1, 2, \dots, k)^e$ for some exponent e. The cycle $(\tau(1), \dots, \tau(k))$ is the same as $(\tau(2), \dots, \tau(k), \tau(1))$ which is the same as $(\tau(i), \tau(i+1), \dots, \tau(k), \tau(1), \dots, \tau(i-1))$ for all i and these are the only sequences $(c_1, \dots, c_k)$ which are equal to $(\tau(1), \dots, \tau(k))$ as k-cycles. This means that $\sigma(1) = \tau(i), \sigma(2) = \tau(i+1), \dots$ for some i and in this case $\sigma = \tau \cdot (1, 2, \dots, k)^{i-1}$.

   (a): A_3 is abelian so all conjugacy classes have one element. For A_4, suppose $(132) = \sigma(123)\sigma^{-1}$. From class we know that RHS is $(\sigma(1), \sigma(2), \sigma(3))$ which immediately gives $\sigma(4) = 4$ as $\sigma(1), \sigma(2), \sigma(3)$ permute $1, 2, 3$ and therefore the sign of σ in S_4 is the same as the sign of σ in S_3. Now the previous claim implies that $\sigma = \tau(1, 2, 3)^e$ for some e where $\tau = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (2, 3)$ so $\varepsilon(\sigma) = \varepsilon(\tau)\varepsilon((1, 2, 3))^e = -1$ which shows that the two 3-cycles cannot be conjugate in A_4.

(b): As in (a) $(1,2,\ldots,n-2,n,n-1) = \sigma(1,2,\ldots,n)\sigma^{-1}$ iff $(\sigma(1),\ldots,\sigma(n)) = (1,2,\ldots,n-2,n,n-1)$. The RHS is the cycle $(\tau(1),\ldots,\tau(n))$ where $\tau = (n-1,n)$ and the claim implies that $\sigma = \tau\cdot(1,2,\ldots,n)^e$ for some e and so $\varepsilon(\sigma) = \varepsilon(\tau)\varepsilon((1,2,\ldots,n))^e = -1$. We used that if n is odd then $(12\ldots n)$ is even.

(c): Note that $(n-2,n-1,n)(1,2,\ldots,n-2,n)(n-2,n-1,n)^{-1} = (1,2,\ldots,n-3,n-1,n-2)$ so if the two cycles are A_n-conjugate then so are $(1,2,\ldots,n-1)$ and $(1,2,\ldots,n-3,n-1,n-2)$ and we already proved in (b) that this is not possible. □

3. Let G be a group. If $g,h \in G$ are two conjugate elements show that there is a bijection between $\{x \in G \mid g = xhx^{-1}\}$ and $\mathrm{Stab}_G(h)$.

   Proof. Suppose $g = aha^{-1}$ with $a \in G$. For $x \in G$ we have $x \in G$ such that $xhx^{-1} = g = aha^{-1}$ iff $a^{-1}xh(a^{-1}x)^{-1} = h$ iff $a^{-1}x \in \mathrm{Stab}_G(h)$ iff $x \in a\,\mathrm{Stab}_G(h)$. The conclusion is immediate. □

4. Let $n \geq 5$ and H a subgroup of S_n. Assume that H is not A_n or S_n. Show that $[S_n : H] \geq n$. [Hint: As in Problem 1-2 on homework 9 look at the homomorphism $S_n \to S_{S_n/H}$.]

   Proof. As in homework 9 we get a homomorphism $S_n \to S_{S_n/H}$ attaching to $\sigma \in S_n$ the permutation of S_n/H given by left multiplication by σ. We know that the kernel of this homomorphism is contained in H. Since ker is a normal subgroup of S_n it has to be 1 or A_n or S_n. The only subgroups of S_n that contain A_n are A_n and S_n and since H is assumed not to be A_n or S_n we conclude that $\ker = 1$ so the homomorphism $S_n \to S_{S_n/H}$ is an injection. Immediately we conclude, simply by comparing cardinalities, that $n! \leq [S_n : H]!$ and so $[S_n : H] \geq n$. □

5. Let p be a prime and G a nonabelian group of order p^3. Show that $[G,G] = Z(G)$.

   Proof. For any G, $Z(G) \lhd G$. For $|G| = p^3$ we know that $Z(G)$ has order a nontrivial power of p from class. As G is not abelian we deduce that $Z(G)$ has order p or p^2 (cannot be all p^3 as then $Z(G) = G$. Now if $Z(G)$ had order p^2 then $G/Z(G)$ would have order p which would have to be cyclic and we showed in class this is not possible. Therefore $Z(G)$ has order p.

   Now $G/Z(G)$ has order p^2 and from class we deduce that $G/Z(G)$ is abelian. From class we deduce that $Z(G)$ contains $[G,G]$. As G is not abelian $[G,G]$ is not trivial. But then $[G,G] \subset Z(G)$ is a nontrivial subgroup of $Z(G) \cong \mathbb{Z}/p\mathbb{Z}$ and since p is a prime we deduce $[G,G] = Z(G)$. □

6. Let $n \geq 3$ be odd. Find all conjugacy classes in the dihedral group D_{2n}.

   Proof. Note that $R^aR^bR^{-a} = R^b$ and $FR^aR^bFR^a = R^{-b}$ so the conjugacy class of R^b is $\{R^b, R^{-b}\}$.

   Now $R^aFR^bR^{-a} = FR^{b-2a}$ while $FR^aFR^bFR^a = FR^{2a-b} = F^{b-2(b-a)}$. We deduce that the conjugacy class of FR^b is $\{FR^{b-2a} \mid a \in \mathbb{Z}\}$.

   If n is odd, R^b is never R^{-b} and multiplication by 2 on $\mathbb{Z}/n\mathbb{Z}$ is surjective so the conjugacy classes are $\{1\}$, $\{R^b, R^{-b}\}$ as b varies from 1 to $(n-1)/2$, and finally $\{F, FR, FR^2, \ldots, FR^{n-1}\}$.

   If n is even then $R^{n/2} = R^{-n/2}$ and we get that the conjugacy classes are $\{1\}$, $\{R^{n/2}\}$, $\{R^b, R^{-b}\}$ for $b = 1,\ldots,(n-2)/2$, $\{F, FR^2, FR^4, \ldots\}$ and $\{FR, FR^3, \ldots\}$. □

7. (a) Show that $\mathrm{PSL}(2,\mathbb{F}_3) := \mathrm{SL}(2,\mathbb{F}_3)/\{\pm I_2\}$ has order 12.

   (b) Show that in $\mathrm{PSL}(2,\mathbb{F}_3)$, $x = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ has order 3, $y = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ and $z = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ have order 2 and commute, and $\mathrm{PSL}(2,\mathbb{F}_3) = \langle x,y,z \rangle$.

   (c) Show that $\mathrm{PSL}(2,\mathbb{F}_3) \cong (\mathbb{Z}/2\mathbb{Z})^2 \rtimes \mathbb{Z}/3\mathbb{Z}$ and conclude that $\mathrm{PSL}(2,\mathbb{F}_3) \cong A_4$. [Hint: Show that $N = \langle y,z \rangle$ is normal in $\mathrm{PSL}(2,\mathbb{F}_3)$. Recall that $\mathrm{GL}(2,\mathbb{Z}/2\mathbb{Z}) \cong S_3$ from a previous homework and show that A_4 is a similar semidirect product that must be isomorphic to this one.]

Proof. (a): The order of $\mathrm{GL}(2, \mathbb{F}_3)$ is $(3^2-1)(3^2-3) = 48$ and $\mathrm{SL}(2, \mathbb{F}_3)$ is the kernel of $\mathrm{GL}(2, \mathbb{F}_3) \to \mathbb{F}_3^\times$ given by determinant which is surjective. The first isomorphism theorem yields that $\mathrm{SL}(2, \mathbb{F}_3)$ has order 24. We quotient by a group of order 2 to get $\mathrm{PSL}(2, \mathbb{F}_3)$ has order 12.

(b): For orders and commutativity simply compute. Now $N = \langle y, z\rangle$ has order 4 and contains two elements of order 2 so it is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Also $H = \langle x\rangle$ has order 3. Since $N \cap H = 1$ by coprimality of orders we deduce that the expressions $\{x^a y^b z^c \mid 0 \le a \le 2, b, c = 0, 1\}$ are all distinct and therefore they exhaust $\mathrm{PSL}(2, \mathbb{F}_3)$.

(c): Since $\mathrm{PSL}(2, \mathbb{F}_3) = HN$ (we don't yet know that N is normal so a priori HN is not a group and we don't know yet that $NH = HN$) to check that N is normal we only need to show that conjugating N by $x^a g$ with $g \in N$ is still N. Therefore we only need to check that $x^a N x^{-a} = N$ and so enough to check that $xNx^{-1} = N$. But $xyx^{-1} = z$ and $xzx^{-1} = yz$ and so $xyzx^{-1} = z$. As $\mathrm{PSL}(2, \mathbb{F}_3) = HN$, $H \cap N = 1$ and N is normal we deduce that $\mathrm{PSL}(2, \mathbb{F}_3) \cong N \rtimes H \cong (\mathbb{Z}/2\mathbb{Z})^2 \rtimes \mathbb{Z}/3\mathbb{Z}$ for some $\phi : \mathbb{Z}/3\mathbb{Z} \to \mathrm{Aut}(N) \cong \mathrm{GL}(2, \mathbb{Z}/2\mathbb{Z}) \cong S_3$. There are exactly 3 such homomorphisms, namely $\phi(1) = 1$, $\phi(1) = (123)$ or $\phi(1) = (123)^{-1}$ in S_3. The trivial homomorphism would yield a direct product but $\mathrm{PSL}(2, \mathbb{F}_3)$ is not commutative. Therefore $\phi(1)$ is either (123) or its inverse. Let's check that both $\phi_\pm(1) = (123)^{\pm 1}$ yield isomorphic semidirect products. Consider the inversion map $a(x) = -x$ on $H \cong \mathbb{Z}/3\mathbb{Z}$. Then a is an automorphism and the fact that ϕ is a homomorphism implies that $\phi_+ = \phi_- \circ a : H \to \mathrm{Aut}(N)$. But under $H \xrightarrow{a} H$ get

$$N \rtimes_{\phi_+} H = N \rtimes_{\phi_+} a(H) \cong N_{\phi_-} \rtimes H$$

and so $\mathrm{PSL}(2, \mathbb{F}_3) \cong N \rtimes H$ where the homomorphism is any of the two nontrivial homomorphisms $H \to \mathrm{Aut}(N)$.

Finally, we know from a previous homework that A_4 has $\{1, (12)(34), (13)(24), (14)(23)\} \cong (\mathbb{Z}/2\mathbb{Z})^2$ as a normal subgroup and $\{(123)\} \cong \mathbb{Z}/3\mathbb{Z}$. Again we get $A_4 \cong (\mathbb{Z}/2\mathbb{Z})^2 \rtimes \mathbb{Z}/3\mathbb{Z}$ and since A_4 is no abelian we deduce that the homomorphism is nontrivial. We already showed that for nontrivial homomorphisms we get isomorphic groups so $\mathrm{PSL}(2, \mathbb{F}_3) \cong A_4$. □

8. (a) Suppose G is a group and $g, h \in G$. Show that gh and hg are conjugate.
   (b) A permutation $\sigma \in S_3$ is said to be good if for every group G and every elements $g_1, g_2, g_3 \in G$, the two products $g_1 g_2 g_3$ and $g_{\sigma(1)} g_{\sigma(2)} g_{\sigma(3)}$ are conjugate in G. Show that σ is good if and only if $\sigma \in \langle (123)\rangle$. [Hint: conjugate matrices have the same trace.]

Proof. (a): $gh = h^{-1} hg (h^{-1})^{-1}$.

(b): If $\sigma \notin A_3$ then σ is a transposition. Let's suppose $\sigma = (23)$ as the others are analogous. Then note that if $g_1 = \begin{pmatrix} 2 & \\ & 1 \end{pmatrix}$, $g_2 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ and $g_3 = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$ then $\mathrm{Tr}(g_1 g_2 g_3) = x$ whereas $\mathrm{Tr}(g_1 g_3 g_2) = 2x$. For $x \neq 0$ we deduce that $g_1 g_2 g_3$ cannot possibly be conjugate to $g_{\sigma(1)} g_{\sigma(2)} g_{\sigma(3)} = g_1 g_3 g_2$. □

9. Artin 7.3.1 on page 222.

Proof. S is a disjoint union of orbits of G so $|S| = \sum |O_s|$. We know that $|O_s| = |G|/|\mathrm{Stab}_G(s)|$ so $|O_s|$ is a power of p. As $|S|$ is not divisible by p, the sum $\sum |O_s|$ is not divisible by p so at least one of the cardinalities $|O_s|$ is $p^0 = 1$. But then $O_s = \{s\} = \{gs \mid g \in G\}$ so s is a fixed point. □

10. Artin 7.5.11 (a) on page 223.

Proof. $C = \{\sigma p \sigma^{-1} \mid \sigma \in S_n\} = \{\sigma p \sigma^{-1} \mid \sigma \in A_n\} \cup \{\sigma p \sigma^{-1} \mid \sigma \notin A_n\}$. But $S_n - A_n = cA_n$ for any odd permutation $c \in S_n$ and so $C = \{\sigma p \sigma^{-1} \mid \sigma \in A_n\} \cup \{\sigma c p c^{-1} \sigma^{-1} \mid \sigma \in A_n\}$. But sets on the RHS are conjugacy classes in A_n so either they are equal or disjoint. We deduce that the S_n conjugacy class

C is either an entire A_n conjugacy class or a disjoint union of 2 of them. It is the former iff $\sigma c p c^{-1} \sigma^{-1}$ is in the first set for some $\sigma \in A_n$. But from Exercise 3 on this problem set, if $\sigma c p c^{-1} \sigma^{-1} = \tau p \tau^{-1}$ then $\tau^{-1} \sigma c \in C_{S_n}(p)$. But if $\sigma, \tau \in A_n$ then $\tau^{-1} \sigma c$ is odd and any odd permutation can be writted as $1^{-1} \sigma c$ for some $\sigma \in A_n$. Therefore the two A_n conjugacy class are the same if and only if $C_{S_n}(p)$ contains odd permutations. $\square$