Math 30810 Honors Algebra 3
Homework 11

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Due at noon on Thursday, November 17

Do 8 of the following questions. Some questions are obligatory. Artin a.b.c means chapter a, section b, exercise c. You may use any problem to solve any other problem.

1. Let p be a prime, $G = \mathrm{GL}(n, \mathbb{F}_p)$ and U the subgroup of upper triangular matrices with 1-s on the diagonal.
   (a) Show that U is a p-Sylow subgroup.
   (b) Show that $N_G(U)$ is the group of upper triangular invertible matrices.
   (c) Determine n_p.
2. Let R be a ring. An idempotent element of R is an element $e \in R$ such that $e^2 = e$. Consider $eR = \{ex \mid x \in R\}$. Show that $(eR, +_R, \cdot_R, 0_R, e)$ is a ring.
3. Show that $\mathbb{C}[x][\![y]\!] \neq \mathbb{C}[\![y]\!][x]$.
4. Suppose R is an integral domain ring with fraction field F (i.e., F is the smallest field containing R as a subring). What is the fraction field of $R[\![x]\!]$.
5. Artin 9.5.6 on page 285. (A one-parameter group is a group of the form $\{e^{tA} \mid t \in \mathbb{R}\}$ where A is a matrix.)
6. Suppose $A \in M_{n\times n}(\mathbb{R})$, $B \in \mathrm{GL}(n, \mathbb{R})$ and $x \in \mathbb{R}$. Recall from a previous homework that exponentials of matrices are always invertible. Compute
$$\frac{d}{dx}[e^{xA}, B]_{\mathrm{GL}(n)}|_{x=0}$$
   where $[\cdot, \cdot]_{\mathrm{GL}(n)}$ is the usual group commutator in the group $\mathrm{GL}(n, \mathbb{R})$.
7. Let $A, B \in M_{n\times n}(\mathbb{R})$ and $x \in \mathbb{R}$. For general matrices write $[A, B] = AB - BA$. Show that
$$e^{xA}e^{xB} = e^{x(A+B)+x^2[A,B]/2+\text{higher order terms}}$$
   [Hint: Write out the first terms of the Taylor series of $\log(e^{xA}e^{xB})$.]
8. Artin 11.1.6 on page 354.
9. Artin 11.1.7 on page 354.

10-11 (Counts as 2 problems) Artin 7.M.12 on page 228.