Math 30810 Honors Algebra 3
Homework 13

Andrei Jorza

Due at noon on Thursday, December 8

Do 7 of the following questions. Some questions are obligatory. Artin a.b.c means chapter a, section b, exercise c. You may use any problem to solve any other problem.

1. (You have to do this problem) Let R be a ring and I an ideal of R. Define $J = \{x \in R \mid x^n \in I \text{ for some } n\}$.
   (a) Show that J is an ideal of R as well.
   (b) What is J when $R = \mathbb{Z}$ and $I = n\mathbb{Z}$ for a positive integer n?

2. Let R be a ring and $N = \{x \in R \mid x^n = 0 \text{ for some } n\}$. The previous problem applied to the 0 ideal shows that N is an ideal of R. Show that N is contained in every prime ideal of R. [Hint: Use the definitions.] (In fact one can show that N equals the intersection of all the prime ideals of R.)

3. (You have to do this problem) Let R be a ring.
   (a) Show that if x is contained in every maximal ideal of R then $1 + xR \subset R^\times$. [Hint: Every proper ideal is contained in some maximal ideal.]
   (b) Show that if $x \in R$ has the property that $1 + xR \subset R^\times$ then x is contained in every maximal ideal of R. [Hint: if $\mathfrak{m}$ is a maximal ideal which doesn't contain x look at $\mathfrak{m} + (x)$.]

4. Suppose R is a ring and $\mathcal{S}$ is an ascending chain of ideals of R, i.e., there exists a totally ordered index set $\mathcal{I}$ such that $\mathcal{S} = \{I_i\}_{i \in \mathcal{I}}$ with $I_i \subset I_j$ whenever $i < j$ in $\mathcal{I}$. Show that $\bigcup_{i \in \mathcal{I}} I_i$ is an ideal of R.

5. Show that $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain. [Hint: Use the complex distance function.]

6-7 (Counts as 2 problems) Let $R = \mathbb{Z}[\sqrt{2}] = \{m + n\sqrt{2} \mid m, n \in \mathbb{Z}\}$ with fraction field $F = \mathbb{Q}(\sqrt{2}) = \{x + y\sqrt{2} \mid x, y \in \mathbb{Q}\}$.
   (a) Show that $d(x + y\sqrt{2}) = |x^2 - 2y^2|$ is multiplicative on $\mathbb{Q}(\sqrt{2})$ and $d(z) = 0$ iff $z = 0$.
   (b) Suppose $a, b \in \mathbb{Z}[\sqrt{2}]$ and write $z = a/b = u + v\sqrt{2} \in \mathbb{Q}(\sqrt{2})$. Let m be the integer closest to u and n be the integer closest to v. Show that if $q = m + n\sqrt{2}$ then $a = bq + r$ for $d(r) < d(b)$ and conclude that $\mathbb{Z}[\sqrt{2}]$ is a Euclidean domain.

8-10 (Counts as 3 problems) Consider the ring $R = \mathbb{Z}[\zeta]$ where $\zeta = e^{2\pi i/3}$.
   (a) Show that if $m^2 + mn + n^2 = 1$ then $m - n\zeta \in \mathbb{Z}[\zeta]^\times$ and if $m^2 + mn + n^2$ is a prime integer then $m - n\zeta$ and $m - n\zeta^2$ are irreducible in $\mathbb{Z}[\zeta]$. [Hint: Use $|\cdot|^2$.]
   (b) Show that $3 = (1 - \zeta)(1 - \zeta^2)$ and $1 - \zeta$ and $1 - \zeta^2$ are irreducible elements of $\mathbb{Z}[\zeta]$.
   (c) Let $p \neq 3$ be a prime integer. Show that if $p \equiv 2 \pmod 3$ then p is irreducible in $\mathbb{Z}[\zeta]$.
   (d) Show that if $p \equiv 1 \pmod 3$ is a prime integer then $p \mid x^2 + x + 1$ for some integer x.
   (e) Deduce that if $p \equiv 1 \pmod 3$ is a prime integer then $p = m^2 + mn + n^2$ for some integers m and n and therefore that $p = (m - n\zeta)(m - n\zeta^2)$ with $m - n\zeta$ and $m - n\zeta^2$ irreducibles in $\mathbb{Z}[\zeta]$.