Math 43900 Problem Solving
Fall 2016
Lecture 10 Calculus

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These problems are taken from the textbook, from Engels' *Problem solving strategies*, from Ravi Vakil's Putnam seminar notes and from Po-Shen Loh's Putnam seminar notes.

1 Overview

Last week we talked about sequences and series as a first topic in calculus. Today we'll concentrate on derivatives and integrals. As I mentioned last week calculus is a vast topic but is also the one where you have seen most examples in your previous courses. You will rarely need any new calculus technique that you haven't seen before. The difficulty, on the contrary, is to patch together all the things you know to obtain a solution.

Again, while cleverness will take you a long way in problem solving calculus, there's no place for being squeamish about algebraic manipulations. Finally, the same comment about rigor applies as last week: when solving problems, don't worry about rigor at first. Better to have a complete solution that's missing steps or perhaps is not as rigorous as it should, than to have a completely rigoros write-up of nothing much.

Calculus problems that you see in competitions, much like in the real world, tend to combine ideas from many topics. You could have a derivative problem for maximization that involves limits of integrals. I identified 3 rough types, although the textbooks has many more collections of calculus related problems in §3:

1. Functions and their analytic properties, e.g., continuity and differentiability.

2. Computing integrals or perhaps limits of integrals.
3. Calculus problems in geometry.

2 Problems

2.1 Continuity and derivatives

Easier

1. Show that every continuous function $f : [a, b] \to [a, b]$ has a fixed point, i.e., $f(c) = c$ for some $c \in [a, b]$. [Hint: the intermediate value theorem.]
2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $|f(x) - f(y)| \geq |x - y|$ for all x, y. Show that (a) f is injective and (b) f is surjective.
3. Solve $2^x = x^2$ for $x > 0$.
4. Determine the largest value of $|z^3 - z + 2|$ as z varies among the complex numbers such that $|z| = 1$. [Hint: use brute force, i.e., write $z = x + iy$ with $x^2 + y^2 = 1$ and reduce to a simple maximization in calc 1 in terms of x. This really is only computations.]

Harder

5. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Show that for every $x \in [0, 1]$ the series $\displaystyle\sum_{n=1}^{\infty} \frac{f(x^n)}{2^n}$ converges. [Hint: At first don't try to be rigorous.]
6. Let $P(X) = a_1 X + a_2 X^2 + \cdots + a_n X^n$ and $Q(X) = \displaystyle\sum_{k=1}^{n} a_k X^k/(2^k - 1)$ with $a_1, \ldots \in \mathbb{R}$. Show that if $Q(2^{n+1}) = Q(1) = 0$ then $P(X)$ has a positive root $< 2^n$. [Hint: $2^k + 2^{2k} + \cdots + 2^{kn} = \dfrac{2^{k(n+1)} - 1}{2^k - 1}$.]
7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Define $g(x) = f(x) \displaystyle\int_0^x f(t)dt$. Show that if $g(x)$ is non-increasing then f is the 0 function. [Hint: Write $g(x)$ as the derivative of a conveniently simple function.]

8. (This is useful. Needs a result from multivariable calculus called the inverse function theorem.) Suppose that in the polynomial $P_t(X) = X^n + a_1(t)X^{n-1} + \cdots + a_{n-1}(t)X + a_n(t)$ the coefficients $a_1(t), \ldots, a_n(t)$ are continuous functions in t. Show that if $P_0(X)$ has no double roots then the roots of $P_t(X)$ also vary continuously in t when t is small. [Hint: Consider the multivariable map f that sends the roots $r_1, \ldots, r_n$ to the coefficients $s_1 = -a_1 = r_1 + \cdots + r_n$, $s_2 = a_2 = \sum r_i r_j$, $\ldots$, $s_n = (-1)^n a_n = \prod r_i$. Compute the determinant $\det(\partial s_i/\partial r_j) = \prod(r_i - r_j)$ by induction. Since the determinant of this derivative is nonzero the result follows.]

2.2 Integrals

Easier

9. Compute $\displaystyle\int \frac{x + \sin x - \cos x - 1}{x + e^x + \sin x} dx$. [Hint: add and subtract e^x on top.]

10. Compute $\displaystyle\int (x^6 + x^3)\sqrt[3]{x^3 + 2}dx$. [Hint: put a factor of x inside the cube root.]

Harder

11. (From this year's VTRMC and also in the textbook exercise 459 and also on the Putnam in 2005) Compute $\displaystyle\int_1^2 \frac{\ln(x)}{2 - 2x + x^2} dx$.

12. Compute
$$\int \frac{x^n}{1 + x + x^2/2! + \cdots + x^n/n!} dx$$
[Hint: What is the derivative of the denominator?]

13. Compute
$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{4n^2 - 1^2}} + \frac{1}{\sqrt{4n^2 - 2^2}} + \cdots + \frac{1}{\sqrt{4n^2 - n^2}} \right)$$
[Hint: Use Riemann sums.]

14. Find all continuous functions $f : \mathbb{R} \to [1, \infty)$ for which there exists $a \in \mathbb{R}$ and $k \in \mathbb{Z}_{\geq 1}$ such that

$$f(x)f(2x)\cdots f(nx) \leq an^k$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}_{\geq 1}$. Show that $f(x) = 1$ for all $x \in \mathbb{R}$. [Hint: Take log and use Riemann sums to estimate $\int_0^1 \ln f(x)dx$.]

2.3 Geometry

Easier

15. Show that every convex polygon can be divided by two perpendicular lines into four regions of equal area.

16. Let $f : [0, \infty) \to \mathbb{R}$ be an increasing continuous function such that $f(0) = 0$. You rotate the graph of $f(x)$ over the interval $[0, a]$ around the y-axis to get the solid R_a, which looks like a dish. Assume that for each a, the volume of R_a is also equal to the volume of water the dish R_a can hold. Find $f(x)$. [Hint: Writing the equality of volumes yields a differential equation that you can solve.]

Harder

17. You slice a sphere with an egg slicer and you get pieces of the same size. Show that the area of these pieces is the same.

18. (From Putnam 2015) Let A and B be points on the hyperbola $xy = 1$. Let P be a point on the chord AB such that the triangle APB has largest area. Show that the area bounded by the hyperbola and the chord AP is the same as the area bounded by the hyperbola and the chord BP.