Math 43900 Problem Solving
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Lecture 14 Brainstorming

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In previous lectures we concentrated on different problem solving topics and techniques, as well as writing up clearly and concisely your solutions. We've also looked into how to choose a problem to work on and when, if at all, to give up. Today we'll concentrate on brainstorming, by which I mean how to poke around effectively when stuck while keeping a clear eye.

1 Problems

1. Given a positive integer n, what is the largest k such that the numbers $1, 2, \dots, n$ can be put into k boxes such that the sum of the numbers in each box is the same? E.g., when $n = 8$ the example $(1, 2, 3, 6)$, $(4, 8)$, $(5, 7)$ shows that the largest k is at least 3.

2. Is there an infinite sequence of real numbers $a_1, a_2, \dots$ such that for every positive integer m one has

$$a_1^m + a_2^m + \cdots = m?$$

3. You are given $\varepsilon > 0$ and two integers h and k. Show that you can find two integers m and n such that

$$\varepsilon < |h\sqrt{m} - k\sqrt{n}| < 2\varepsilon$$

4. Let $\mathcal{S}$ be a class of functions from $[0, \infty)$ to $[0, \infty)$ that satisfies:
   (a) The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x + 1)$ are in $\mathcal{S}$;
   (b) If $f(x), g(x)$ are in $\mathcal{S}$ then so are the function $f(x) + g(x)$ and $f(g(x))$;
   (c) If $f(x), g(x)$ are in $\mathcal{S}$ and $f(x) \geq g(x)$ for all $x \geq 0$ then the function $f(x) - g(x)$ is in $\mathcal{S}$. Prove that if $f(x), g(x)$ are in $\mathcal{S}$ then so is the function $f(x)g(x)$.

5. Let A be the $n \times n$ matrix whose entry on row i and column j is $1/\min(i, j)$. Compute $\det A$.

6. Find the volume of the region of points (x, y, z) such that

$$(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2)$$